Optimal CDMA Spreading Code Allocation

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Abstract

An important multiple-access technique in wireless networks and other common channel communication systems is Code-Division Multiple Access (CDMA). Each user shares the entire bandwidth with all the other users and is distinguished from the others by its signature sequence or code. Each user spreads its information on the common channel through modulation using its signature sequence. Then the receiver demodulates the transmitted messages upon observing the sum of the transmitted signals embedded in noise. We focus on symbol-synchronous CDMA (S-CDMA) systems where in each symbol interval the received signal is the sum of the transmitted signals in that symbol interval alone embedded in additive white Gaussian noise.

It is already known in literature [8] that the sum capacity of the symbol-synchronous code-division multiple-access channel with equal average-input-energy constraints is maximized precisely by those spreading sequence multisets that meet Welch's lower bound on total squared correlation. For a general case of asymmetric user power constraints, the signature sequences with real components that achieve sum capacity are also identified [9].

However, the assumption of real signature sequences is impractical. In this thesis, we will consider binary signature sequences. The sum capacities of Walsh Code, m-sequences and Binary Almost Perfect Sequences (BAPS) will be evaluated

respectively. In addition, an optimal code allocation scheme will also be proposed for an ad-hoc based S-CDMA system.

Besides, another important measure for the performance of a CDMA system is the Signal to Interference Ratio (SIR). SIR would degrade significantly due to the mobility of users. A simple code adaptation scheme (Simplified Maximum Collision Time) for a mobile CDMA system is proposed to combat the effect of mobility. The scheme is simulated and the performance is evaluated.

摘要

分碼多重存取(CDMA)在無線網絡和其他共用通訊系統中，是非常重要的一種多重存取技術。每個用戶與所有其他用戶共同分享整個頻譜，而每個用戶則被分派不同的序列碼(signature sequence)， 以作識別。在調制過程中，用戶會先利用序列碼去擴展其原訊息。之後，接收器在收到所有用戶訊號加上噪音之中進行解調。我們將集中討論符碼同步分碼多重存取(symbol-synchronous CDMA)系統。在每一個符碼間區，接收的訊號是所有發出的訊號及附加白高斯噪音(AWGN)之和。

在相同平均輸入能量限制下，那些序列碼多集合符合韋爾相關平方總和之下限(Welch's lower bound on total squared correlation)，可以最大化符碼同步分碼多重存取頻道的容量和(sum capacity) [8]。更進一步，在非對稱用戶功率限制下，那些能夠最大化容量和的實(real)序列碼亦已被找到 [9]。

但是，使用實序列碼的假設並不現實。在這份論文中，我們會著重考慮二進序列碼。沃爾甚碼(Walsh Code)、最大長度序碼(m-sequence)和二進近乎完美序列(binary almost perfect sequence)的容量和會分別地計算出來。再者，對於 ad-hoc 同步分碼多重存取系統，我們亦提出一個最佳的序碼分配方案。

另一方面，評價一個分碼多重存取系統優劣的另一個重要標準是訊號干擾比(SIR)。用戶的流動性(mobility)會明顯地削弱訊號干擾比。為了對抗用戶流動性的不良影響，我們提議了一個簡單，有效的序碼調節方案。這個方案會被模擬，其效益亦被評價。

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Chapter 1

Introduction

There are in practice two approaches to the problem of multiple access schemes. One is the FDMA/TDMA cellular approach and the other is the spread spectrum approach. The cellular approach is to divide up the geographical area into cells each with a centrally located base station and the mobiles in a cell communicate with the base station in their cell. This idea is common to both schemes in practice although it is possible to extend it so that a mobile near a cell boundary can communicate with two or more nearby receivers.

The fundamental difference between the schemes above is the following: in the FDMA/TDMA approach the frequency spectrum is divided up into subchannels and a given user occupying one operates at a high signal to noise ratio (C/I). To achieve this high C/I no other user in the same cell as the given user or in an adjacent cell to that user can reuse this subchannel. However, the subchannel can be reused in a cell sufficiently far away since signal strengths fade with distance. Consequently, the full bandwidth is partitioned over clusters of cells; those adjacent to each other do not use the same frequency bands. The spread spectrum approach is to use the whole bandwidth in each cell; in fact, to allow each mobile to use the whole bandwidth. The users' signals are not orthogonal, the C/I is low and strong error correcting codes are necessary. As the C/I is low in spread spectrum there is much to be gained from diversity reception; it is beneficial for users near cell boundaries to communicate with all the nearby base stations. It may even be appropriate to dispense with the idea of cells altogether.

There is clearly a limit to the number of users that a given scheme can accommodate, the limit imposed by the bandwidth available and the interference between users. Current approaches to spread spectrum suffer the "near far" problem. The "near far" problem refers to the situation where one user's received power is high relative to another, for example when the interfering user is closer to the receiver. Received power fluctuations compound the problem. Power control can be used to reduce fluctuations but the capacity is limited by interference. In FDMA/TDMA, the interference problem is mitigated through reuse partitioning but this means capacity is limited by bandwidth. As the reuse distance is based on a worst case scenario and the variance of power fluctuations is high, capacity is wasted much of the time.

Mobile radio is a complex subject and hence difficult to model accurately. Rather than trying to incorporate all factors (slow fading, fast fading, multipath …) we take a simple model which we now outline. It is important to note that we deal only with the mobile to base station link. We assume that the mobiles have fixed locations and the received powers at the single receiver do not fluctuate. This model is the basis for chapter 2. In chapter 3, we remove the constraint of fixed locations and incorporate the mobility factor into our model. We use another approach to evaluate the performance of the single-cell CDMA System under a simple code adaptation scheme.

We examine the communication model from a general point of view. We consider it as a multi-user communication channel consisting of a single or multiple receivers. The theory of information then gives the capacity of this channel and we interpret the result to see what type of transmission scheme is appropriate. There may appear to be an inconsistency between the information-theoretic notion of capacity and the notion that is relevant of mobile radio. In mobile radio each user has data rate R and the system is at capacity when the number of users cannot be increased. Multi-user information theory deals with a fixed number of users and makes statements about how high the individual rates can be; the outcome is a feasible rate region. With K users, one obtains feasible $(R_1, R_2, \cdots, R_K)$ where R_i is the bit rate of user i. The region is given as the convex closure of the intersection of various regions bounded by hyper planes each arising from a constraint on information corresponding to the interaction of a particular subset of the K users. However, if we require these rates to be the same, then the system is at capacity when the common rate R cannot be increased. Both notions of capacity then coincide.

Clearly, if we are sufficiently below capacity, then there is no problem at all; there are enough orthogonal codes available and these can be allocated to the users. It is equally clear that codes do not always have to be orthogonal. If two users are sufficiently far apart in a large mobile network then they can use the same code; this is the basis of the cellular reuse approach. However, when we are at capacity it may be necessary to have non-orthogonality even though interference results.

Outline of Thesis

In Chapter 2, we examine the capacity of a general synchronous DS-CDMA System. Many previous works [8][9] have been focused on maximizing the capacity by choosing the best signature sequences set. However, the results assumed that the signature sequences had real components (as opposed to components in {+1, -1}). We will focus on binary sequence only and evaluate the capacity for several special sequence sets like, Walsh Code, m-sequence, and Binary Almost Perfect Sequence. The asymptotic upper bounds for those capacities would also be derived. Besides, an optimal code allocation scheme is proposed for an ad-hoc S-CDMA system.

In Chapter 3, we analyze the same system but take a totally different approach. The evaluation criterion is the signal to interference ratio (SIR) rather than the capacity of the system. We allow the sender to move and as time goes by, the delay shift between any pair of senders would be changed. This would degrade the signal to interference ratio as the cross-correlation between the spreading codes varies. A simple code adaptation scheme is proposed and the performance in combating the mobility effect would be evaluated.

Chapter 2

Capacity of Single-Cell S-CDMA System

In this chapter, we will state the results for a Single-Cell S-CDMA system. Section 2.1 provides the entire necessary preliminary and forms the basis for the following sections. In Section 2.2, the sum capacities of 3 special sequence sets are evaluated. In Section 2.3, the asymptotic upper bound of the sum capacity for 2 sequence sets is presented. Finally, an optimal dynamic code allocation scheme for ad-hoc S-CDMA system is proposed in Section 2.4.

2.1 Preliminary

2.1.1 Information Measure

Information theory answers two fundamental questions in communication theory: what is the ultimate data compression, and what is the ultimate transmission rate of communication. The answer is related to the amount of information which is abstract. So, we need to have a measure of information.

ENTROPY

Entropy is a measure of *uncertainty* of a random variable. Let X be a discrete random variable with alphabet χ and probability mass function $p(x) = \Pr\{X = x\}$, $x \in \chi$.

Definition: The *entropy* $H(X)$ of a discrete random variable X is defined by

$$H(X) = -\sum_{x \in \chi} p(x) \log p(x) \tag{2.1}$$

The log is to the base 2 and entropy is expressed in bits. Note that the entropy $H(X)$ of a random variable X is a functional of the probability distribution $p(x)$ which measures the average amount of information contained in X, or equivalently, the amount of uncertainty removed upon revealing the outcome of X. It depends on $p(x)$ only, not on the actual values taken by the random variable X.

Example: Let

$$X = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } 1-p \end{cases}$$

Then, $H(X) = -p\log p - (1-p)\log(1-p)$

The entropy of a binary random variable with success probability p, is commonly represented by $H(p)$ where $H(p) = -p\log p - (1-p)\log(1-p)$.

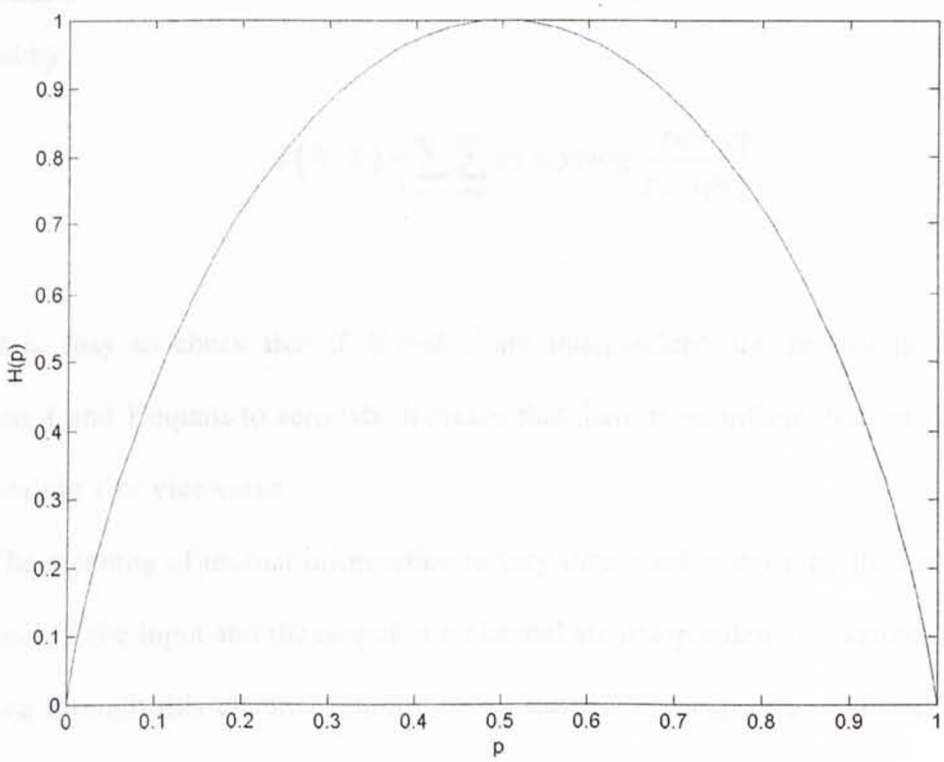

Figure 2.1: The value of entropy of a binary random variable

The figure illustrates some of the basic properties of entropy; it equals to 0 when $p = 0$ or 1. This is obvious that when $p = 0$ or 1, there is no uncertainty since the random variable is not random at all. The entropy achieves the maximum value 1 when $p = 1/2$ which corresponds to the maximum uncertainty.

MUTUAL INFORMATION

Entropy only measures the uncertainty of random variable(s). Sometimes, we want to measure the dependency between two or more random variables, i.e. the amount of information that one random variable contains about another random variable. It is the reduction in the uncertainty of one random variable due to the knowledge of the other.

Definition: For random variables X and Y, the *mutual information* between X and Y is defined by

$$I(X;Y) = \sum_{x \in \chi} \sum_{y \in \psi} p(x,y) \log \frac{p(x,y)}{p(x)p(y)} \tag{2.2}$$

It is easy to check that if X and Y are independent, the mutual information between X and Y equals to zero which means that there is no information gained for X by revealing Y or vice versa.

The meaning of mutual information is very important in defining the capacity of a channel. If the input and the output of a channel are independent, we cannot transmit anything through this channel reliably by all means. The capacity of this channel is actually zero. We will see later that the capacity of a channel is merely the mutual information between the input and output of the channel.

The definitions of entropy and mutual information above are for discrete random variables only. There are corresponding definitions for continuous random variables and given below.

Definition: Let X be a continuous random variable with probability density function $f(x)$. The set where $f(x)>0$ is called the *support set* of *X*. The *differential entropy* $h(X)$ is defined as

$$h(X)=-\int_S f(x)\log f(x)dx \tag{2.3}$$

where S is the support set of the random variable X.

Definition: The mutual information $I(X;Y)$ between two continuous random variables with joint density $f(x,y)$ is defined as

$$I(X;Y)=-\int f(x,y)\log\frac{f(x,y)}{f(x)f(y)}dxdy \tag{2.4}$$

Example: Let X be a zero mean, normal random variable with variance σ^2, i.e. $X\sim N\left(0,\sigma^2\right)$. Then, $f(x)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{x^2}{2\sigma^2}}$, the differential entropy is:

$$\begin{aligned}
h(X)&=-\int f(x)\log f(x)dx\\
&=-\log e\cdot\int f(x)\ln f(x)dx\\
&=-\log e\cdot\int f(x)\left[-\frac{x^2}{2\sigma^2}-\ln\left(\sqrt{2\pi\sigma^2}\right)\right]dx\\
&=\log e\cdot\left[\frac{EX^2}{2\sigma^2}+\frac{1}{2}\ln\left(2\pi\sigma^2\right)\right]\\
&=\log e\cdot\left[\frac{1}{2}+\frac{1}{2}\ln\left(2\pi\sigma^2\right)\right]\\
&=\log e\cdot\frac{1}{2}\ln\left(2\pi e\sigma^2\right)\\
&=\frac{1}{2}\log\left(2\pi e\sigma^2\right)
\end{aligned}$$

2.1.2 Channel Capacity

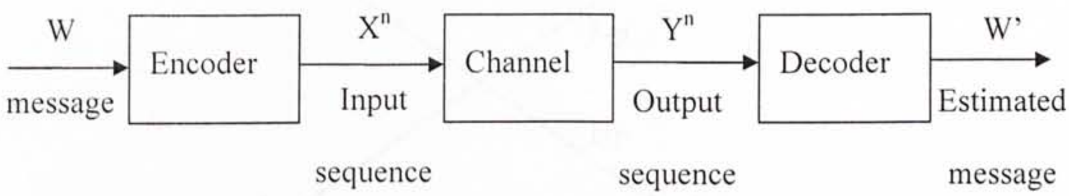

Figure 2.2: A communication system

A model of communication system is shown in Figure 2.2. Source symbols from some finite alphabet are mapped into some sequence of channel symbols, which then produces the output sequence of the channel. The output sequence is random but has a distribution that depends on the input sequence. From the output sequence, we attempt to recover the transmitted message.

Suppose we wish to send one of L equally likely messages down a noisy channel. As there is noise in the channel there is always a chance of making a decoding error and clearly this chance can be reduced by adding redundancy to the encoding. The question of interest is how large L can be, or more precisely, how many bits per second can be transmitted?

It can be shown that there is a capacity C such that information can be transmitted arbitrarily reliably at rate less than C but not at rates greater than C.

Definition: The *"information" channel capacity* of a discrete memoryless channel is:

$$C = \max_{p(x)} I(X;Y) \tag{2.5}$$

where the maximum is taken over all possible input distribution $p(x)$.

Example (Binary Symmetric Channel):

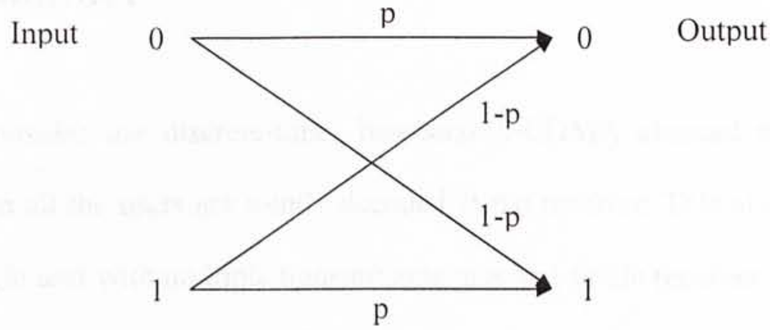

Figure 2.3: Symbol diagram of Binary Symmetric Channel

Consider the binary symmetric channel (BSC), which is shown in Figure 2.3. When a 0 is received as a 1 or a 1 is received as a 0, an error occurs. The mutual information between the input symbol and output symbol is:

$$\begin{aligned} I(X;Y) &= H(Y) - H(Y \mid X) \\ &= H(Y) - \sum p(x) H(Y \mid X = x) \\ &= H(Y) - \sum p(x) H(p) \\ &= H(Y) - H(p) \\ &\leq 1 - H(p) \end{aligned}$$

where $H(p)$ is the entropy of a binary random variable of success probability p and the last inequality follows because Y is a binary random variable. Equality is achieved when the input distribution is uniform. Hence the information capacity of a binary symmetric channel with parameter p is

$$C = 1 - H(p) \quad bits$$

2.1.3 Capacity of multiple access Gaussian DS-CDMA channel

We consider the discrete-time, baseband S-CDMA channel model [8]. The signals from all the users are jointly decoded at the receiver. This also applies to the case of single user with multiple transmit antennas and single receiver.

Assume there are K users in the system and the processing gain is L. We model the information transmitted (symbol) by each user as independent random variables $X_1, X_2, \cdots, X_K$. We assume that there is an average input power constraint on the transmit symbols such that

$$E[X_i^{\,2}] \le p_i \qquad \forall i = 1,2,\cdots K \tag{2.6}$$

Let $\mathbf{P} = diag(p_1, p_2, \cdots, p_K)$ and the maximum total average input power be $p_{tot} = \sum_{i=1}^{K} p_i$. Let the signature sequence of user i is represented by $\underline{\mathbf{s}}_i$, a vector in $\Re^L$. Each signature sequence has power equal to L, i.e. for each user i, we have $\underline{\mathbf{s}}_i^{\,T} \cdot \underline{\mathbf{s}}_i = L$. We assume an AWGN $\underline{\mathbf{N}} = [N_1, N_2, \cdots, N_L]^T$ presents where $\underline{\mathbf{N}}$ is a zero-mean, Gaussian vector with covariance matrix $E[\underline{\mathbf{N}}\underline{\mathbf{N}}^T] = N_o I_L$.

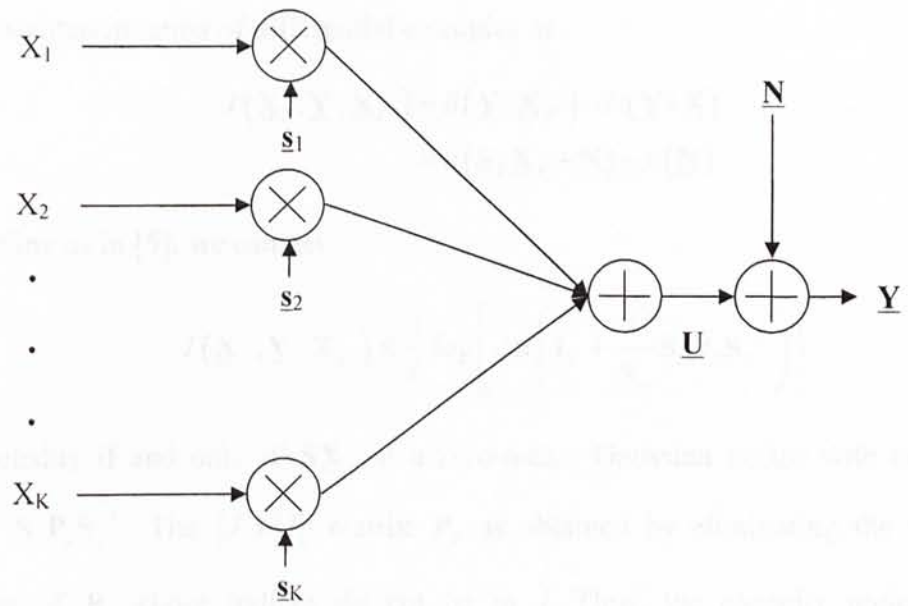

Figure 2.4: Discrete-time, baseband S-CDMA channel model

Then output of the multiple access channel, represented by $\underline{\mathbf{Y}}$ as shown in Figure 2.4 can be written as

$$\underline{\mathbf{Y}} = \sum_{i=1}^{K} \underline{\mathbf{s}}_i X_i + \underline{\mathbf{N}} \tag{2.7}$$

Let $\mathbf{S}$ be a $L \times K$ matrix whose k-th column is $\underline{\mathbf{s}}_k$. Then $\underline{\mathbf{Y}}$ can be rewritten as

$$\underline{\mathbf{Y}} = \mathbf{S}\underline{\mathbf{X}} + \underline{\mathbf{N}} \tag{2.8}$$

where $\underline{\mathbf{X}} = [X_1, X_2, \cdots, X_K]^T$.

For this channel, the capacity region is the closure of the convex hull of the union over all product probability densities $p_{\underline{\mathbf{X}}}$ on the inputs $X_1, X_2, \cdots, X_K$ of the rate regions

$$C(\mathbf{S}, p_{\underline{\mathbf{X}}}) = \bigcap_{\substack{J \subseteq \{1,2,\cdots,K\} \\ J \neq \{\,\}}} \left\{ (R_1, R_2, \cdots, R_K) : 0 \le \sum_{i \in J} R_i \le I(\underline{\mathbf{X}}_J ; \underline{\mathbf{Y}} \mid \underline{\mathbf{X}}_{J^c}) \right\} \tag{2.9}$$

where J is a non-null subset and J^c is the complement. R_i is the rate in bits per chip of user i. The vectors $\underline{\mathbf{X}}_J$ and $\underline{\mathbf{X}}_{J^c}$ are obtained by eliminating the components of $\underline{\mathbf{X}}$

whose indices belong to J and J^c, respectively. The conditional mutual information can be written in terms of differential entropies as

$$\begin{aligned} I(\underline{\mathbf{X}}_J;\underline{\mathbf{Y}}\mid\underline{\mathbf{X}}_{J^c}) &= h(\underline{\mathbf{Y}}\mid\underline{\mathbf{X}}_{J^c}) - h(\underline{\mathbf{Y}}\mid\underline{\mathbf{X}}) \\ &= h(\mathbf{S}_J\underline{\mathbf{X}}_J+\underline{\mathbf{N}}) - h(\underline{\mathbf{N}}) \end{aligned}$$

Proceeding as in [5], we can get

$$I(\underline{\mathbf{X}}_J;\underline{\mathbf{Y}}\mid\underline{\mathbf{X}}_{J^c}) \le \frac{1}{L}\log\left[\det\left(\mathbf{I}_L+\frac{1}{N_o}\mathbf{S}_J\mathbf{P}_J\mathbf{S}_J{}^T\right)\right] \tag{2.10}$$

with equality if and only if $\mathbf{S}\underline{\mathbf{X}}_J$ is a zero-mean, Gaussian vector with covariance matrix $\mathbf{S}_J\mathbf{P}_J\mathbf{S}_J{}^T$. The $|J|\times|J|$ matrix $\mathbf{P}_J$ is obtained by eliminating the rows and columns of $\mathbf{P}$ whose indices do not lie in J. Thus, the capacity region of the S-CDMA channel is given by

$$C(\mathbf{S}) = \bigcap_{\substack{J\subseteq\{1,2,\cdots,K\}\\ J\neq\{\ \}}} \left\{(R_1,R_2,\cdots,R_K): 0\le\sum_{i\in J}R_i\le\frac{1}{L}\log\left[\det\left(\mathbf{I}_L+\frac{1}{N_o}\mathbf{S}_J\mathbf{P}_J\mathbf{S}_J{}^T\right)\right]\right\}$$

and is strongly depends on the sequence matrix $\mathbf{S}$.

The resultant sum capacity which is a reasonable criterion of goodness for the sequence matrix is defined by

$$C_{SUM}(\mathbf{S}) = \max_{(R_1,R_2,\cdots,R_K)\in C(\mathbf{S})}\sum_{k=1}^{K}R_k \tag{2.11}$$

It follows from above and the choice $J=\{1,2,\cdots,K\}$ that

$$C_{sum}(\mathbf{S}) = \frac{1}{L}\log\left[\det\left(\mathbf{I}_L+\frac{1}{N_o}\mathbf{SPS}^T\right)\right] \tag{2.12}$$

The upper bound of $C_{sum}(\mathbf{S})$ has been shown in [9] which is

$$C_{sum}(\mathbf{S}) \le \log\left[1+\frac{p_{tot}}{N_o}\right] \tag{2.13}$$

For both cases of equal average input power constraint and unequal average input power constraint for all users, the sequence multiset $\mathbf{S}$ that achieves the upper bound for a given average input power constraint **P** has been found in [8] and [9], respectively. However, for the case of unequal average input power constraint which is more practical, the resultant sequence multiset takes on real components instead of binary components e.g. {+1, -1}. Practically, signature sequences take binary components only. Thus, we will investigate the sum capacity of binary sequence multiset in the following section, though it is not guaranteed that there always exists a binary sequence multiset that can achieve the upper bound of the sum capacity.

2.2 Evaluation of the sum capacity for selected sequence sets

As shown before, for a Gaussian multiple-access S-CDMA channel with K users, the ith user has an average input power constraint on the transmit symbols such that

$$E[X_i^2] \le p_i$$

and the signature sequence is $\underline{\mathbf{s}}_i$ whose length is L.

The sum capacity of this channel can be found to be:

$$C_{sum}(\mathbf{S}) = \frac{1}{L}\log\left[\det\left(\mathbf{I}_L + \frac{1}{N_o}\mathbf{SPS}^T\right)\right]$$

where $\mathbf{P} = diag(p_1, p_2, \cdots, p_K)$ and $\mathbf{S} = [\underline{\mathbf{s}}_1, \underline{\mathbf{s}}_2, \ldots, \underline{\mathbf{s}}_K]$.

Further let $\mathbf{W} = diag(w_1, w_2, \cdots, w_K)$ where $w_k = \dfrac{p_k}{N_o}$. We have,

$$C_{sum}(\mathbf{S}) = \frac{1}{L}\log\left[\det\left(\mathbf{I}_L + \mathbf{SWS}^T\right)\right] \tag{2.14}$$

Consider:

$$\begin{aligned}
\det\left(\mathbf{I}_L + \mathbf{SWS}^T\right) &= \det\left(\mathbf{I}_L + \mathbf{SW}^{\frac{1}{2}}\mathbf{W}^{\frac{1}{2}}\mathbf{S}^T\right) \\
&= \det\left(\mathbf{I}_K + \mathbf{W}^{\frac{1}{2}}\mathbf{S}^T\mathbf{SW}^{\frac{1}{2}}\right) \\
&= \det\left(\mathbf{W}^{\frac{1}{2}}\mathbf{W}^{-1}\mathbf{W}^{\frac{1}{2}} + \mathbf{W}^{\frac{1}{2}}\mathbf{S}^T\mathbf{SW}^{\frac{1}{2}}\right) \\
&= \det\left(\mathbf{W}^{\frac{1}{2}}\left(\mathbf{W}^{-1} + \mathbf{S}^T\mathbf{S}\right)\mathbf{W}^{\frac{1}{2}}\right) \\
&= \left[\det\left(\mathbf{W}^{\frac{1}{2}}\right)\right]^2 \det\left(\mathbf{W}^{-1} + \mathbf{S}^T\mathbf{S}\right)
\end{aligned}$$

where

$$\mathbf{W}^{\frac{1}{2}} = diag\left(\sqrt{w_1}, \sqrt{w_2}, \cdots, \sqrt{w_K}\right)$$

$$\mathbf{W}^{-1} = diag\left(\frac{1}{w_1}, \frac{1}{w_2}, \cdots, \frac{1}{w_K}\right)$$

Therefore, we have

$$C_{sum}(\mathbf{S}) = \frac{1}{L}\log\left[\left[\det\left(\mathbf{W}^{\frac{1}{2}}\right)\right]^2 \det\left(\mathbf{W}^{-1} + \mathbf{S}^T\mathbf{S}\right)\right] \tag{2.15}$$

$\det\left(\mathbf{W}^{\frac{1}{2}}\right)$ is easy to found and

$$\left[\det\left(\mathbf{W}^{\frac{1}{2}}\right)\right]^2 = \prod_{i=1}^{K}\frac{p_i}{N_o} \tag{2.16}$$

which is independent of the signature sequences $\mathbf{S}$.

What is remaining to find is $\det\left(\mathbf{W}^{-1} + \mathbf{S}^T\mathbf{S}\right)$ and is generally difficult to evaluate for arbitrary signature sequences set. We will consider the case for Walsh Codes, m-sequence and Binary Almost Perfect Sequence in the following subsections.

2.2.1 Walsh Code

Walsh Codes are obtained by selecting as codewords the rows of a *Hadamard matrix* which is a type of square (+1, -1)-matrix. A Hadamard matrix **A** is a $L \times L$ matrix of +1s and -1s such that each row differs from any other row in exactly $L/2$ locations. One row contains all 1s with the remainder containing $L/2$ 1s and $L/2$ -1s. The minimum distance for these codes is $L/2$.

Hadamard matrix can be generated recursively. For instance,

$$\mathbf{H}_2 = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$\mathbf{H}_4 = \begin{bmatrix} \mathbf{H}_2 & \mathbf{H}_2 \\ \mathbf{H}_2 & -\mathbf{H}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 \end{bmatrix}$$

$$\mathbf{H}_8 = \begin{bmatrix} \mathbf{H}_4 & \mathbf{H}_4 \\ \mathbf{H}_4 & -\mathbf{H}_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \end{bmatrix}$$

The set of Walsh Code of length L, $L = 2^m$ for $m > 1$, contains L mutually orthogonal sequences. Let $\underline{\mathbf{s}}_1, \underline{\mathbf{s}}_2, \cdots, \underline{\mathbf{s}}_L$ be those sequences. We assume that all users are assigned one of those sequences and it is possible that more than one user are assigned to the same sequence. As a result, the correlation of sequences between any pair of users is either zero or L.

Further partition the index of user, from 1 to K, into L sets $S_1, S_2, \cdots, S_L$, such that if the i-th user is assigned the sequence $\underline{\mathbf{s}}_j$, then the element i would belong to the set S_j. Some sets may be empty or contain more than one element. Let l be the number of non-empty sets of $S_1, S_2, \cdots, S_L$, without loss of generality, let the first l sequences $\underline{\mathbf{s}}_1, \underline{\mathbf{s}}_2, \cdots, \underline{\mathbf{s}}_l$ are being assigned to one or more users and the remaining L-l sequences $\underline{\mathbf{s}}_l, \underline{\mathbf{s}}_{l+1}, \cdots, \underline{\mathbf{s}}_L$ are not being used by anyone.

Example 2.1 for Walsh Code:

$K = 7$, $L = 4$. $\underline{\mathbf{s}}_1$ is assigned to user 1 while $\underline{\mathbf{s}}_2$ is assigned to user 2,5 and $\underline{\mathbf{s}}_3$ is assigned to user 3,4,6,7. Then, $S_1 = \{1\}$, $S_2 = \{2,5\}$, $S_3 = \{3,4,6,7\}$, $S_4 = \{\ \}$ and $l = 3$. Further let $P_{(j)}$ be the sum of power of those users using $\underline{\mathbf{s}}_j$ i.e. $P_{(j)} = \sum_{i \in S_j} p_i$ Then, $P_{(1)} = p_1$, $P_{(2)} = p_2 + p_5$, $P_{(3)} = p_3 + p_4 + p_6 + p_7$ and $P_{(4)} = 0$. The matrix $\mathbf{W}^{-1} + \mathbf{S}^T\mathbf{S}$ for this example is:

$$\mathbf{W}^{-1} + \mathbf{S}^T\mathbf{S} = \begin{bmatrix} \frac{N_o}{p_1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{N_o}{p_2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{N_o}{p_3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{N_o}{p_4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{N_o}{p_5} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{N_o}{p_6} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{N_o}{p_7} \end{bmatrix} + \begin{bmatrix} L & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & L & 0 & 0 & L & 0 & 0 \\ 0 & 0 & L & L & 0 & L & L \\ 0 & 0 & L & L & 0 & L & L \\ 0 & L & 0 & 0 & L & 0 & 0 \\ 0 & 0 & L & L & 0 & L & L \\ 0 & 0 & L & L & 0 & L & L \end{bmatrix}$$

$$
=\begin{bmatrix}
L+\frac{N_o}{p_1} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & L+\frac{N_o}{p_2} & 0 & 0 & L & 0 & 0 \\
0 & 0 & L+\frac{N_o}{p_3} & L & 0 & L & L \\
0 & 0 & L & L+\frac{N_o}{p_4} & 0 & L & L \\
0 & L & 0 & 0 & L+\frac{N_o}{p_5} & 0 & 0 \\
0 & 0 & L & L & 0 & L+\frac{N_o}{p_6} & L \\
0 & 0 & L & L & 0 & L & L+\frac{N_o}{p_7}
\end{bmatrix}
$$

Since the index of user is arbitrary, by suitable rearrangement of rows and columns.

$$
\mathbf{W}^{-1}+\mathbf{S}^T\mathbf{S}=\begin{bmatrix}
L+\frac{N_o}{p_1} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & L+\frac{N_o}{p_2} & L & 0 & 0 & 0 & 0 \\
0 & L & L+\frac{N_o}{p_5} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & L+\frac{N_o}{p_3} & L & L & L \\
0 & 0 & 0 & L & L+\frac{N_o}{p_4} & L & L \\
0 & 0 & 0 & L & L & L+\frac{N_o}{p_6} & L \\
0 & 0 & 0 & L & L & L & L+\frac{N_o}{p_7}
\end{bmatrix}
$$

We can rewrite the matrix as:

$$\mathbf{W}^{-1} + \mathbf{S}^T\mathbf{S} = diag(\mathbf{D}_1, \mathbf{D}_2, \ldots, \mathbf{D}_l)$$

where

$$\mathbf{D}_j = \begin{bmatrix} L + \frac{N_o}{p.} & L & \ldots & L \\ L & L + \frac{N_o}{p.} & & L \\ \vdots & & \ddots & \vdots \\ L & L & \ldots & L + \frac{N_o}{p.} \end{bmatrix}$$

is a $|S_j| \times |S_j|$ square matrix whose diagonal elements contain the power of user $i, \forall i \in S_j$.

Proposition 1:

Let $C_{SUM}(Walsh)$ be the sum capacity of the multiple access Gaussian DS-CDMA channel employing Walsh Code. Then

$$C_{SUM}(Walsh) = \frac{1}{L}\log\left\{\prod_{j=1}^{l}\left[1 + \frac{LP_{(j)}}{N_o}\right]\right\} \tag{2.17}$$

where l is the number of sequence being assigned to at least one user.

Before proving proposition 1, we need one lemma.

Definition: Let $\mathbf{A}_k(r)$ be a $k \times k$ matrix whose elements are all r. Let $\underline{\mathbf{c}}_k(r)$ be a $k \times 1$ column vector whose elements are all r. i.e.

$$\mathbf{A}_k(r) = \begin{bmatrix} r & r & \ldots & r \\ r & r & & r \\ \vdots & & \ddots & \vdots \\ r & r & \ldots & r \end{bmatrix} \qquad \underline{\mathbf{c}}_k(r) = \begin{bmatrix} r \\ r \\ \vdots \\ r \end{bmatrix}$$

Lemma 1:

Let $\mathbf{D}$ be a $j \times j$ diagonal matrix, such that

$$\mathbf{D} = \begin{bmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & a_j \end{bmatrix}$$

Then

$$\det\left(\mathbf{D} + \mathbf{A}_j(r)\right) = \det\begin{bmatrix} r + a_1 & r & \dots & r \\ r & r + a_2 & & r \\ \vdots & & \ddots & \vdots \\ r & r & \dots & r + a_j \end{bmatrix}$$
$$= r\prod_{i=1}^{j} a_i \left[\frac{1}{r} + \sum_{i=1}^{j}\frac{1}{a_i}\right]$$

The proof of Lemma 1 is left in Appendix.

Proof of Proposition 1:

$$C_{SUM}(Walsh) = \frac{1}{L}\log\left[\left[\det\left(\mathbf{W}^{\frac{1}{2}}\right)\right]^2 \det\left(\mathbf{W}^{-1} + \mathbf{S}^T\mathbf{S}\right)\right]$$

where $\left[\det\left(\mathbf{W}^{\frac{1}{2}}\right)\right]^2 = \prod_{i=1}^{K}\frac{p_i}{N_o}$ and

$$\det\left(\mathbf{W}^{-1} + \mathbf{S}^T\mathbf{S}\right) = \det\left(diag\left(\mathbf{D}_1, \mathbf{D}_2, \dots, \mathbf{D}_l\right)\right)$$
$$= \prod_{k=1}^{l}\det\left(\mathbf{D}_k\right)$$
$$= \prod_{k=1}^{l}\left[L\prod_{i\in S_k}\frac{N_o}{p_i}\left[\frac{1}{L} + \sum_{i\in S_k}\frac{p_i}{N_o}\right]\right]$$
$$= L^l\left(\prod_{i=1}^{K}\frac{N_o}{p_i}\right)\prod_{k=1}^{l}\left[\frac{1}{L} + \sum_{i\in S_k}\frac{p_i}{N_o}\right]$$

Therefore,

$$
\begin{aligned}
C_{SUM}\left(Walsh\right) &= \frac{1}{L}\log\left\{\left(\prod_{i=1}^{K}\frac{p_i}{N_o}\right)\left[L^l\left(\prod_{i=1}^{K}\frac{N_o}{p_i}\right)\prod_{k=1}^{l}\left[\frac{1}{L}+\sum_{i\in S_k}\frac{p_i}{N_o}\right]\right]\right\} \\
&= \frac{1}{L}\log\left\{L^l\prod_{k=1}^{l}\left[\frac{1}{L}+\sum_{i\in S_k}\frac{p_i}{N_o}\right]\right\} \\
&= \frac{1}{L}\log\left\{\prod_{j=1}^{l}\left[1+\frac{LP_{(j)}}{N_o}\right]\right\}
\end{aligned}
$$

2.2.2 m-sequence

A *pseudo-noise (PN) sequence* is a periodic binary sequence with a noise-like waveform that is usually generated by means of a *feedback shift register*. A feedback shift register consists of an ordinary *shift register* made up of m flip-flops and a *logic circuit* that are interconnected to form a multiloop *feedback* circuit. The flip-flops in the shift register are regulated by a single timing clock. At each pulse of the clock, the *state* of each flip-flop is shifted to the next one down the line. With each clock pulse the logic circuit computes a Boolean function of the states of the flip-flops. The result is then fed back as the input to the first flip-flop, thereby preventing the shift register from emptying. The PN sequence so generated is determined by the length m of the shift register, its initial state, and the feedback logic.

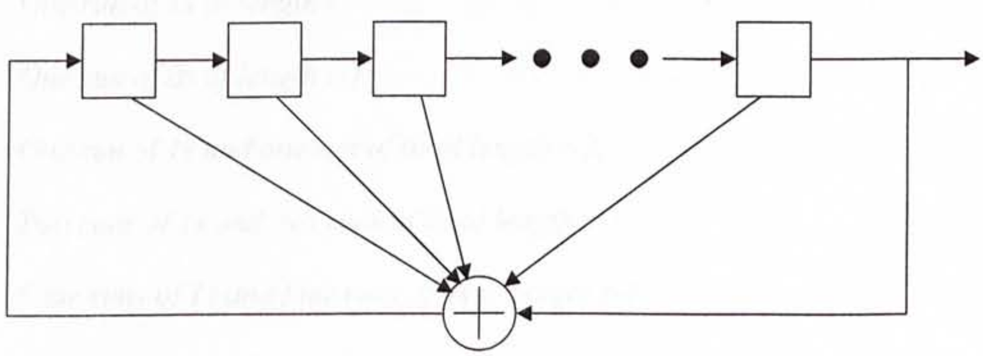

Figure 2.5: Construction of linear feedback shift register sequence

If the feedback logic is *linear*, the PN sequence will be called *linear feedback shift register sequence*, a block diagram of which is shown in Figure 2.5. Furthermore, if the length of the sequence is exactly $L = 2^m - 1$, the PN sequence is called a *maximal-length-sequence* or simply *m-sequence*.

m-sequences have a number of properties that are useful in their application to spread-spectrum systems. Some of these properties are given here.

PROPERTY 1: *A m-sequence contains one more one than zero. The number of ones in the sequence is* $\frac{1}{2}(L+1)$.

PROPERTY 2: *The modulo-2 sum of an m-sequence and any phase shift of the same sequence is another phase of the same m-sequences (shift-and-add property).*

PROPERTY 3: *If a window of width r is slid along the sequence for L shifts, each r-tuple except the all-zero r-tuple will appear exactly once.*

PROPERTY 4: *Define a run as a subsequence of identical symbols within the m-sequence. The length of this subsequence is the length of the run. Then, for any m-sequence, there are*

1. *One run of 1s of length r.*
2. *One run of 0s of length r-1.*
3. *One run of 1s and one run of 0s of length r-2.*
4. *Two runs of 1s and two runs of 0s of length r-3.*
5. *Four runs of 1s and four runs of 0s of length r-4.*

$\vdots$

r. 2^{r-3} *runs of 1s and* 2^{r-3} *runs of 0s of length 1.*

PROPERTY 5: *After the mapping of 0→1, 1→-1, the periodic autocorrelation function is 2-valued and is given by*

$$R_x(\tau) = \begin{cases} L & \tau = kL \\ -1 & \tau \neq kL \end{cases}$$

where k is an integer value, and τ *is the relative shift as a multiple of a chip-period.*

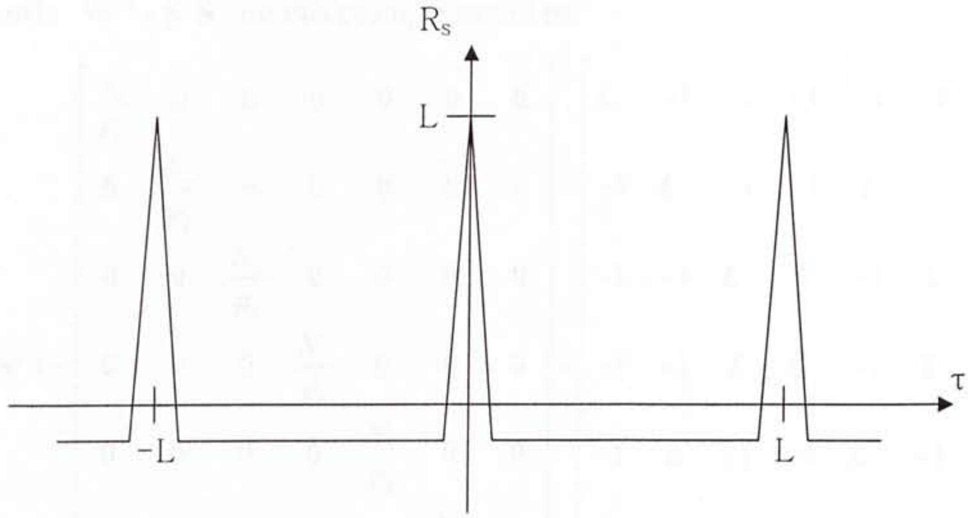

Figure 2.6: Periodic autocorrelation of m-sequence

Unlike the Walsh Code whose codewords are chosen directly from a matrix, the set of sequences for m-sequence is formed by first choosing a m-sequence $\underline{\mathbf{s}}$ of length L, $L = 2^m - 1$ for $m > 1$. Then the set of sequences $\underline{\mathbf{s}}_1, \underline{\mathbf{s}}_2, \cdots, \underline{\mathbf{s}}_L$ correspond to all the L phases of $\underline{\mathbf{s}}$.

We assume that all users are assigned one of those sequences and it is possible that more than one user are assigned to the same sequence. As a result, by PROPERTY 5, the correlation of sequences between any pair of users is either *-1* or *L*.

Further partition the index of user, from 1 to *K*, into *L* sets $S_1, S_2, \cdots, S_L$, such that if the *i*-th user is assigned the sequence $\underline{\mathbf{s}}_j$, then the element *i* would belong to the set S_j. Some sets may be empty or contain more than one element. Let *l* be the number of non-empty sets of $S_1, S_2, \cdots, S_L$, without loss of generality, let the first *l* sequences $\underline{\mathbf{s}}_1, \underline{\mathbf{s}}_2, \cdots, \underline{\mathbf{s}}_l$ are being assigned to one or more users and the remaining *L-l* sequences $\underline{\mathbf{s}}_l, \underline{\mathbf{s}}_{l+1}, \cdots, \underline{\mathbf{s}}_L$ are not being used by anyone.

Example 2.1 for m-sequence:

The matrix $\mathbf{W}^{-1}+\mathbf{S}^T\mathbf{S}$ for this example would be:

$$\mathbf{W}^{-1}+\mathbf{S}^T\mathbf{S}=\begin{bmatrix} \frac{N_o}{p_1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{N_o}{p_2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{N_o}{p_3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{N_o}{p_4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{N_o}{p_5} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{N_o}{p_6} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{N_o}{p_7} \end{bmatrix}+\begin{bmatrix} L & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & L & -1 & -1 & L & -1 & -1 \\ -1 & -1 & L & L & -1 & L & L \\ -1 & -1 & L & L & -1 & L & L \\ -1 & L & -1 & -1 & L & -1 & -1 \\ -1 & -1 & L & L & -1 & L & L \\ -1 & -1 & L & L & -1 & L & L \end{bmatrix}$$

$$\mathbf{W}^{-1}+\mathbf{S}^T\mathbf{S}=\begin{bmatrix} L+\frac{N_o}{p_1} & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & L+\frac{N_o}{p_2} & -1 & -1 & L & -1 & -1 \\ -1 & -1 & L+\frac{N_o}{p_3} & L & -1 & L & L \\ -1 & -1 & L & L+\frac{N_o}{p_4} & -1 & L & L \\ -1 & L & -1 & -1 & L+\frac{N_o}{p_5} & -1 & -1 \\ -1 & -1 & L & L & -1 & L+\frac{N_o}{p_6} & L \\ -1 & -1 & L & L & -1 & L & L+\frac{N_o}{p_7} \end{bmatrix}$$

By suitable rearrangement of rows and columns.

$$
\mathbf{W}^{-1} + \mathbf{S}^T\mathbf{S} = \begin{bmatrix}
L+\frac{N_o}{p_1} & -1 & -1 & -1 & -1 & -1 & -1 \\
-1 & L+\frac{N_o}{p_2} & L & -1 & -1 & -1 & -1 \\
-1 & L & L+\frac{N_o}{p_5} & -1 & -1 & -1 & -1 \\
-1 & -1 & -1 & L+\frac{N_o}{p_3} & L & L & L \\
-1 & -1 & -1 & L & L+\frac{N_o}{p_4} & L & L \\
-1 & -1 & -1 & L & L & L+\frac{N_o}{p_6} & L \\
-1 & -1 & -1 & L & L & L & L+\frac{N_o}{p_7}
\end{bmatrix}
$$

We can rewrite the matrix as:

$$
\mathbf{W}^{-1} + \mathbf{S}^T\mathbf{S} = \bar{\mathbf{D}} + \mathbf{A}_K(-1)
$$

where

$$
\bar{\mathbf{D}} = diag\left(\bar{\mathbf{D}}_1, \bar{\mathbf{D}}_2, \ldots, \bar{\mathbf{D}}_l\right)
$$

$$
\bar{\mathbf{D}}_l = \begin{bmatrix}
L+1+\frac{N_o}{p.} & L+1 & \ldots & L+1 \\
L+1 & L+1+\frac{N_o}{p.} & & L+1 \\
\vdots & & \ddots & \vdots \\
L+1 & L+1 & \ldots & L+1+\frac{N_o}{p.}
\end{bmatrix}
\qquad
\mathbf{A}_K(-1) = \begin{bmatrix}
-1 & -1 & \ldots & -1 \\
-1 & -1 & & -1 \\
\vdots & & \ddots & \vdots \\
-1 & -1 & \ldots & -1
\end{bmatrix}
$$

$\bar{\mathbf{D}}_l$ is a $|S_j| \times |S_j|$ square matrix whose diagonal elements contain the power of user $i, \forall i \in S_j$.

Proposition 2:

Let $C_{SUM}(m-seq)$ be the sum capacity of the multiple access Gaussian DS-CDMA channel employing m-sequence. Then

$$C_{SUM}(m-seq)=\frac{1}{L}\log\left\{\prod_{j=1}^{l}\left[1+\frac{(L+1)P_{(j)}}{N_o}\right]\cdot\left[1-\sum_{i=1}^{l}\frac{P_{(i)}}{N_o+(L+1)P_{(i)}}\right]\right\} \quad (2.18)$$

where l is the number of sequence being assigned to at least one user.

Before proving proposition 2, we need one more lemma.

Lemma 2:

Let $\mathbf{H}$ be a $n\times n$ square matrix, define the operation $SUM(\mathbf{H})$ be the sum of all n^2 elements. Let $\mathbf{D}$ be a $j\times j$ diagonal matrix, such that

$$\mathbf{D}=\begin{bmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & a_j \end{bmatrix}$$

Then,

$$SUM\left[\left(\mathbf{D}+\mathbf{A}_j(r)\right)^{-1}\right]=SUM\left(\begin{bmatrix} r+a_1 & r & \dots & r \\ r & r+a_2 & & r \\ \vdots & & \ddots & \vdots \\ r & r & \dots & r+a_j \end{bmatrix}^{-1}\right)$$

$$=\frac{\sum_{i=1}^{j}\frac{1}{a_i}}{1+r\sum_{i=1}^{j}\frac{1}{a_i}}$$

The proof is left in Appendix.

Proof of Proposition 2:

$$C_{SUM}(m-seq)=\frac{1}{L}\log\left[\left[\det\left(\mathbf{W}^{\frac{1}{2}}\right)\right]^2\det\left(\mathbf{W}^{-1}+\mathbf{S}^T\mathbf{S}\right)\right]$$

where $\left[\det\left(W^{\frac{1}{2}}\right)\right]^2=\prod_{i=1}^{K}\frac{p_i}{N_o}$ and

$$\begin{aligned}
\det\left(\mathbf{W}^{-1}+\mathbf{S}^T\mathbf{S}\right) &= \det\left(\tilde{\mathbf{D}}+\mathbf{A}_K(-1)\right)\\
&= \det\left(\tilde{\mathbf{D}}+\underline{\mathbf{c}}_K(-1)\cdot\underline{\mathbf{c}}_K(1)^T\right)\\
&= \det\left[\tilde{\mathbf{D}}\left(\mathbf{I}+\tilde{\mathbf{D}}^{-1}\underline{\mathbf{c}}_K(-1)\cdot\underline{\mathbf{c}}_K(1)^T\right)\right]\\
&= \det\left(\tilde{\mathbf{D}}\right)\det\left(\mathbf{I}+\tilde{\mathbf{D}}^{-1}\underline{\mathbf{c}}_K(-1)\cdot\underline{\mathbf{c}}_K(1)^T\right)\\
&= \det\left(\tilde{\mathbf{D}}\right)\left(1+\underline{\mathbf{c}}_K(1)^T\cdot\tilde{\mathbf{D}}^{-1}\underline{\mathbf{c}}_K(-1)\right)\\
&= \det\left(\tilde{\mathbf{D}}\right)\left[1-SUM\left(\tilde{\mathbf{D}}^{-1}\right)\right]\\
&= \det\left(\tilde{\mathbf{D}}\right)\left[1-\sum_{i=1}^{I}SUM\left(\tilde{\mathbf{D}}_i^{-1}\right)\right]
\end{aligned}$$

by lemma 2,

$$SUM\left(\tilde{\mathbf{D}}_i^{-1}\right)=\frac{\sum_{m\in S_i}\frac{p_m}{N_o}}{1+(L+1)\sum_{m\in S_i}\frac{p_m}{N_o}}=\frac{P_{(i)}}{N_o+(L+1)P_{(i)}}$$

and

$$\begin{aligned}
\det\left(\tilde{\mathbf{D}}\right) &= \prod_{j=1}^{I}\det\left(\tilde{\mathbf{D}}_j\right)\\
&= \prod_{j=1}^{I}\left[(L+1)\prod_{i\in O_k}\frac{N_o}{p_i}\left[\frac{1}{(L+1)}+\sum_{i\in O_k}\frac{p_i}{N_o}\right]\right]\\
&= (L+1)^I\left(\prod_{i=1}^{K}\frac{N_o}{p_i}\right)\prod_{j=1}^{I}\left[\frac{1}{(L+1)}+\frac{P_{(j)}}{N_o}\right]
\end{aligned}$$

Therefore,

$$
\begin{aligned}
& C_{SUM}\left(m-seq\right) \\
&= \frac{1}{L}\log\left\{\left(\prod_{i=1}^{K}\frac{p_i}{N_o}\right)\left\{\left(L+1\right)^{l}\left(\prod_{i=1}^{K}\frac{N_o}{p_i}\right)\prod_{j=1}^{l}\left[\frac{1}{\left(L+1\right)}+\frac{P_{(j)}}{N_o}\right]\right\}\left(1-\sum_{i=1}^{l}\frac{P_{(i)}}{N_o+\left(L+1\right)P_{(i)}}\right)\right\} \\
&= \frac{1}{L}\log\left\{\prod_{j=1}^{l}\left[1+\frac{\left(L+1\right)P_{(j)}}{N_o}\right]\cdot\left(1-\sum_{i=1}^{l}\frac{P_{(i)}}{N_o+\left(L+1\right)P_{(i)}}\right)\right\}
\end{aligned}
$$

2.2.3 Binary Almost Perfect Sequence

A binary sequence of period L is called *perfect* if it has a 2-level autocorrelation function where the off-peak autocorrelation coefficients γ are as small as theoretically possible (in absolute value). It is not at all clear that perfect sequences exist, and it can be shown that for many values of L, no such sequences can exist [11][13].

Due to difficulty in searching for binary perfect sequences with smallest off-peak autocorrelation, research in searching for "Binary Almost Perfect Sequences" is motivated. A binary almost perfect sequence is a binary sequence where all the off-peak autocorrelation coefficients are as small as theoretically possible – with exactly one exception. Of course, it would be useful if the exceptional value is also small, but this is not required.

For instance, binary almost perfect sequences with the following three-valued autocorrelation were proposed [11]:

$$R_s(\tau) = \sum_{i=1}^{L} b_i b_{i+\tau} = \begin{cases} L, & \tau = 0(\bmod L) \\ -(L-4), & \tau = L/2 \\ 0, & else \end{cases}$$

where the period L of these sequences is a multiple of 4.

In the evaluation of the sum capacity of binary almost perfect sequence, we make use of the above example. However, we generalize the exceptional off-peak autocorrelation value to be any value. That is, the three-valued autocorrelation function is:

$$R_s(\tau) = \sum_{i=1}^{L} b_i b_{i+\tau} = \begin{cases} L, & \tau = 0(\bmod L) \\ R, & \tau = L/2 \\ 0, & else \end{cases}$$

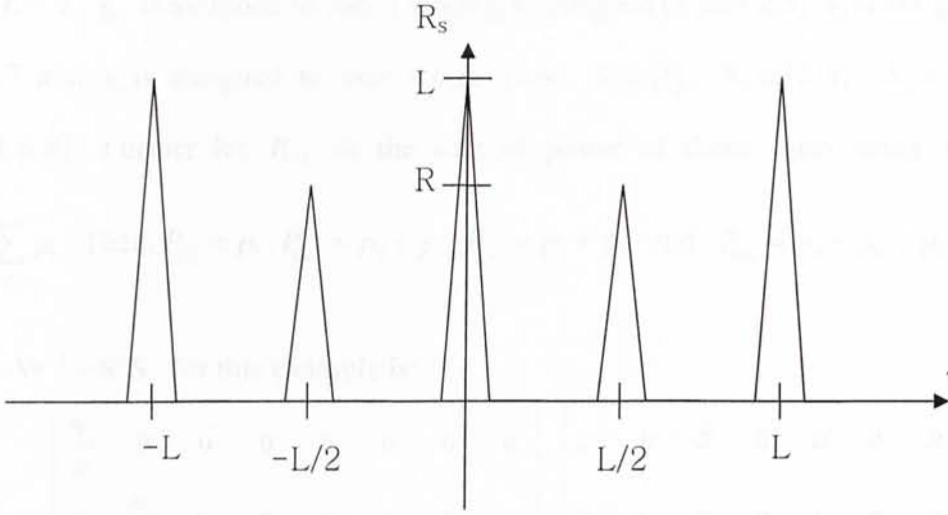

Figure 2.7: Periodic autocorrelation of BAPS

Similar to m-sequence, the set of sequences for BAPS is formed by first choosing a BAPS $\underline{\mathbf{s}}$ of length L. Then the set of sequences $\underline{\mathbf{s}}_1, \underline{\mathbf{s}}_2, \cdots, \underline{\mathbf{s}}_L$ correspond to all the L phases of $\underline{\mathbf{s}}$.

We assume that all users are assigned one of those sequences and it is possible that more than one user are assigned to the same sequence. As a result, the correlation of sequences between any pair of users is either *-1*, *R* or *L*.

Further partition the index of user, from 1 to *K*, into *L* sets $S_1, S_2, \cdots, S_L$, such that if the *i*-th user is assigned the sequence $\underline{\mathbf{s}}_j$, then the element *i* would belong to the set S_j. Some sets may be empty or contain more than one element. But for simplicity, we assume that each sequence is being assigned to at least one user.

Example 2.2:

$K = 8$, $L = 4$. $\underline{\mathbf{s}}_1$ is assigned to user 1 while $\underline{\mathbf{s}}_2$ is assigned to user 2,5, $\underline{\mathbf{s}}_3$ is assigned to user 3,7 and $\underline{\mathbf{s}}_4$ is assigned to user 4,6,8. Then, $S_1 = \{1\}$, $S_2 = \{2,5\}$, $S_3 = \{3,7\}$, $S_4 = \{4,6,8\}$. Further let $P_{(j)}$ be the sum of power of those users using $\underline{\mathbf{s}}_j$ i.e. $P_{(j)} = \sum_{i \in S_j} p_i$ Then, $P_{(1)} = p_1$, $P_{(2)} = p_2 + p_5$, $P_{(3)} = p_3 + p_7$ and $P_{(4)} = p_4 + p_6 + p_8$. The matrix $\mathbf{W}^{-1} + \mathbf{S}^T\mathbf{S}$ for this example is:

$$\mathbf{W}^{-1} + \mathbf{S}^T\mathbf{S} = \begin{bmatrix} \frac{N_o}{p_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{N_o}{p_2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{N_o}{p_3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{N_o}{p_4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{N_o}{p_5} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{N_o}{p_6} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{N_o}{p_7} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{N_o}{p_8} \end{bmatrix} + \begin{bmatrix} L & 0 & R & 0 & 0 & 0 & R & 0 \\ 0 & L & 0 & R & L & R & 0 & R \\ R & 0 & L & 0 & 0 & 0 & L & 0 \\ 0 & R & 0 & L & R & L & 0 & L \\ 0 & L & 0 & R & L & R & 0 & R \\ 0 & R & 0 & L & R & L & 0 & L \\ R & 0 & L & 0 & 0 & 0 & L & 0 \\ 0 & R & 0 & L & R & L & 0 & L \end{bmatrix}$$

$$\mathbf{W}^{-1}+\mathbf{S}^{T}\mathbf{S}=\begin{bmatrix} L+\frac{N_o}{p_1} & 0 & R & 0 & 0 & 0 & R & 0 \\ 0 & L+\frac{N_o}{p_2} & 0 & R & L & R & 0 & R \\ R & 0 & L+\frac{N_o}{p_3} & 0 & 0 & 0 & L & 0 \\ 0 & R & 0 & L+\frac{N_o}{p_4} & R & L & 0 & L \\ 0 & L & 0 & R & L+\frac{N_o}{p_5} & R & 0 & R \\ 0 & R & 0 & L & R & L+\frac{N_o}{p_6} & 0 & L \\ R & 0 & L & 0 & 0 & 0 & L+\frac{N_o}{p_7} & 0 \\ 0 & R & 0 & L & R & L & 0 & L+\frac{N_o}{p_8} \end{bmatrix}$$

By suitable rearrangement of rows and columns.

$$\mathbf{W}^{-1}+\mathbf{S}^{T}\mathbf{S}=\begin{bmatrix} L+\frac{N_o}{p_1} & 0 & 0 & R & R & 0 & 0 & 0 \\ 0 & L+\frac{N_o}{p_2} & L & 0 & 0 & R & R & R \\ 0 & L & L+\frac{N_o}{p_5} & 0 & 0 & R & R & R \\ R & 0 & 0 & L+\frac{N_o}{p_3} & L & 0 & 0 & 0 \\ R & 0 & 0 & L & L+\frac{N_o}{p_7} & 0 & 0 & 0 \\ 0 & R & R & 0 & 0 & L+\frac{N_o}{p_4} & L & L \\ 0 & R & R & 0 & 0 & L & L+\frac{N_o}{p_6} & L \\ 0 & R & R & 0 & 0 & L & L & L+\frac{N_o}{p_8} \end{bmatrix}$$

We can rewrite the matrix as:

$$
\mathbf{W}^{-1}+\mathbf{S}^T\mathbf{S}=\begin{pmatrix}
\boxed{\mathbf{D}_1} & 0 & \cdots & 0 & \boxed{\mathbf{R}_1} & 0 & \cdots & 0 \\
0 & \boxed{\mathbf{D}_2} & & 0 & 0 & \boxed{\mathbf{R}_2} & & 0 \\
\vdots & & \ddots & \vdots & \vdots & & \ddots & \vdots \\
0 & 0 & \cdots & \boxed{\mathbf{D}_{L/2}} & 0 & 0 & \cdots & \boxed{\mathbf{R}_{L/2}} \\
\boxed{\mathbf{R}_1^T} & 0 & \cdots & 0 & \boxed{\mathbf{D}_{L/2+1}} & 0 & \cdots & 0 \\
0 & \boxed{\mathbf{R}_2^T} & & 0 & 0 & \boxed{\mathbf{D}_{L/2+2}} & & 0 \\
\vdots & & \ddots & \vdots & \vdots & & \ddots & \vdots \\
0 & 0 & \cdots & \boxed{\mathbf{R}_{L/2}^T} & 0 & 0 & \cdots & \boxed{\mathbf{D}_L}
\end{pmatrix}
$$

where

$$
\mathbf{D}_l=\begin{bmatrix}
L+\frac{N_o}{p.} & L & \cdots & L \\
L & L+\frac{N_o}{p.} & & L \\
\vdots & & \ddots & \vdots \\
L & L & \cdots & L+\frac{N_o}{p.}
\end{bmatrix}
\qquad
\mathbf{R}_j=\begin{bmatrix}
R & R & \cdots & R \\
R & R & & R \\
\vdots & & \ddots & \vdots \\
R & R & \cdots & R
\end{bmatrix}
$$

$\mathbf{D}_l$ is a $|S_l|\times|S_l|$ square matrix whose diagonal elements contain the power of user $i, \forall i \in S_l$.

$\mathbf{R}_j$ is a $|S_j|\times|S_{L/2+j}|$ matrix whose elements are all equal to R, for $j=1,2,\cdots,L/2$.

We can further divide the matrix into 4 parts:

$$
\mathbf{W}^{-1}+\mathbf{S}^T\mathbf{S}=\left(\begin{array}{cccc|cccc}
\boxed{\mathbf{D}_1} & 0 & \cdots & 0 & \boxed{\mathbf{R}_1} & 0 & \cdots & 0 \\
0 & \boxed{\mathbf{D}_2} & & 0 & 0 & \boxed{\mathbf{R}_2} & & 0 \\
\vdots & & \ddots & \vdots & \vdots & & \ddots & \vdots \\
0 & 0 & \cdots & \boxed{\mathbf{D}_{L/2}} & 0 & 0 & \cdots & \boxed{\mathbf{R}_{L/2}} \\
\hline
\boxed{\mathbf{R}_1^T} & 0 & \cdots & 0 & \boxed{\mathbf{D}_{L/2+1}} & 0 & \cdots & 0 \\
0 & \boxed{\mathbf{R}_2^T} & & 0 & 0 & \boxed{\mathbf{D}_{L/2+2}} & & 0 \\
\vdots & & \ddots & \vdots & \vdots & & \ddots & \vdots \\
0 & 0 & \cdots & \boxed{\mathbf{R}_{L/2}^T} & 0 & 0 & \cdots & \boxed{\mathbf{D}_L}
\end{array}\right)
$$

$$\mathbf{W}^{-1}+\mathbf{S}^T\mathbf{S}=\left[\begin{array}{c:c}\mathbf{A}_1 & \mathbf{A}_2 \\ \hdashline \mathbf{A}_2^T & \mathbf{A}_3\end{array}\right]$$

where

$$\mathbf{A}_1 = diag\left(\mathbf{D}_1,\mathbf{D}_2,\ldots,\mathbf{D}_{L/2}\right)$$
$$\mathbf{A}_2 = diag\left(\mathbf{R}_1,\mathbf{R}_2,\ldots,\mathbf{R}_{L/2}\right)$$
$$\mathbf{A}_3 = diag\left(\mathbf{D}_{L/2+1},\mathbf{D}_{L/2+2},\ldots,\mathbf{D}_L\right)$$

Proposition 3:

Let $C_{SUM}\left(BAPS\right)$ be the sum capacity of the multiple access Gaussian DS-CDMA channel employing binary almost perfect sequence. Then

$$C_{SUM}\left(BAPS\right)=\frac{1}{L}\log\left\{\prod_{i=1}^{L/2}\left(1+\frac{LP_{(L/2+i)}}{N_o}-\frac{R^2P_{(i)}P_{(L/2+i)}}{N_o\left(N_o+LP_{(i)}\right)}\right)\left(1+\frac{LP_{(i)}}{N_o}\right)\right\} \quad (2.19)$$

Proof of Proposition 3:

$$C_{SUM}\left(BAPS\right)=\frac{1}{L}\log\left[\left[\det\left(\mathbf{W}^{\frac{1}{2}}\right)\right]^2\det\left(\mathbf{W}^{-1}+\mathbf{S}^T\mathbf{S}\right)\right]$$

where $\left[\det\left(\mathbf{W}^{\frac{1}{2}}\right)\right]^2=\prod_{j=1}^{K}\frac{p_j}{N_o}$ and

$$\det\left(\mathbf{W}^{-1}+\mathbf{S}^T\mathbf{S}\right)=\det\begin{pmatrix}\mathbf{A}_1 & \mathbf{A}_2\\ \mathbf{A}_2^T & \mathbf{A}_3\end{pmatrix}$$
$$=\det\left(\mathbf{A}_1\right)\det\left(\mathbf{A}_3-\mathbf{A}_2^T\mathbf{A}_1^{-1}\mathbf{A}_2\right)$$

where $\mathbf{A}_1^{-1}=diag\left(\mathbf{D}_1^{-1},\mathbf{D}_2^{-1},\ldots,\mathbf{D}_{L/2}^{-1}\right)$

Consider $\mathbf{A}_2^T\mathbf{A}_1^{-1}\mathbf{A}_2$

$$\mathbf{A}_2^T\mathbf{A}_1^{-1}\mathbf{A}_2$$
$$=diag\left(\mathbf{R}_1^T,\mathbf{R}_2^T,\ldots,\mathbf{R}_{L/2}^T\right)\cdot diag\left(\mathbf{D}_1^{-1},\mathbf{D}_2^{-1},\ldots,\mathbf{D}_{L/2}^{-1}\right)\cdot diag\left(\mathbf{R}_1,\mathbf{R}_2,\ldots,\mathbf{R}_{L/2}\right)$$
$$=diag\left(\mathbf{R}_1^T\mathbf{D}_1^{-1}\mathbf{R}_1,\mathbf{R}_2^T\mathbf{D}_2^{-1}\mathbf{R}_2,\ldots,\mathbf{R}_{L/2}^T\mathbf{D}_{L/2}^{-1}\mathbf{R}_{L/2}\right)$$

$$\text{since} \quad \mathbf{R}_i^T \mathbf{D}_i^{-1} \mathbf{R}_i = \underline{\mathbf{c}}_{|S_{L/2+i}|}(R)\underline{\mathbf{c}}_{|S_i|}(1)^T \mathbf{D}_i^{-1} \underline{\mathbf{c}}_{|S_i|}(1)\underline{\mathbf{c}}_{|S_{L/2+i}|}(R)^T$$

$$= \underline{\mathbf{c}}_{|S_{L/2+i}|}(R) SUM\left(\mathbf{D}_i^{-1}\right) \underline{\mathbf{c}}_{|S_{L/2+i}|}(R)^T$$

$$= SUM\left(\mathbf{D}_i^{-1}\right) \mathbf{A}_{|S_{L/2+i}|}\left(R^2\right)$$

$$= \mathbf{A}_{|S_{L/2+i}|}\left(R^2 SUM\left(\mathbf{D}_i^{-1}\right)\right)$$

Thus,

$$\mathbf{A}_2^T \mathbf{A}_1^{-1} \mathbf{A}_2$$

$$= diag\left(\mathbf{A}_{|S_{L/2+1}|}\left(R^2 SUM\left(\mathbf{D}_1^{-1}\right)\right), \mathbf{A}_{|S_{L/2+2}|}\left(R^2 SUM\left(\mathbf{D}_2^{-1}\right)\right), \ldots, \mathbf{A}_{|S_L|}\left(R^2 SUM\left(\mathbf{D}_{L/2}^{-1}\right)\right)\right)$$

and

$$\mathbf{A}_3 - \mathbf{A}_2^T \mathbf{A}_1^{-1} \mathbf{A}_2$$

$$= diag\left(\mathbf{D}_{L/2+1} - \mathbf{A}_{|S_{L/2+1}|}\left(R^2 SUM\left(\mathbf{D}_1^{-1}\right)\right), \mathbf{D}_{L/2+2} - \mathbf{A}_{|S_{L/2+2}|}\left(R^2 SUM\left(\mathbf{D}_2^{-1}\right)\right), \ldots, \right.$$

$$\left. \mathbf{D}_L - \mathbf{A}_{|S_L|}\left(R^2 SUM\left(\mathbf{D}_{L/2}^{-1}\right)\right)\right)$$

since

$$\mathbf{D}_j = \begin{bmatrix} L + \frac{N_o}{p.} & L & \ldots & L \\ L & L + \frac{N_o}{p.} & & L \\ \vdots & & \ddots & \vdots \\ L & L & \ldots & L + \frac{N_o}{p.} \end{bmatrix} = \tilde{\mathbf{D}}_j + A_{|S_j|}(L)$$

$$\textit{where} \quad \tilde{\mathbf{D}}_j = \begin{bmatrix} \frac{N_o}{p.} & 0 & \ldots & 0 \\ 0 & \frac{N_o}{p.} & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \ldots & \frac{N_o}{p.} \end{bmatrix}$$

$$\mathbf{D}_{L/2+i} - \mathbf{A}_{|S_{L/2+i}|}\left(R^2 SUM\left(\mathbf{D}_i^{-1}\right)\right) = \tilde{\mathbf{D}}_{L/2+i} + \mathbf{A}_{|S_{L/2+i}|}(L) - \mathbf{A}_{|S_{L/2+i}|}\left(R^2 SUM\left(\mathbf{D}_i^{-1}\right)\right)$$

$$= \tilde{\mathbf{D}}_{L/2+i} + \mathbf{A}_{|S_{L/2+i}|}\left(L - R^2 SUM\left(\mathbf{D}_i^{-1}\right)\right)$$

Let $K_i = L - R^2 SUM\left(\mathbf{D}_i^{-1}\right)$

then

$$\mathbf{A}_3 - \mathbf{A}_2{}^T\mathbf{A}_1{}^{-1}\mathbf{A}_2 = diag\left(\tilde{\mathbf{D}}_{L/2+1} + \mathbf{A}_{|S_{L/2+1}|}(K_1), \tilde{\mathbf{D}}_{L/2+2} + \mathbf{A}_{|S_{L/2+2}|}(K_2), \ldots, \tilde{\mathbf{D}}_L + \mathbf{A}_{|S_L|}(K_{L/2})\right)$$

$$\begin{aligned}
&\det\left(\mathbf{A}_3 - \mathbf{A}_2{}^T\mathbf{A}_1{}^{-1}\mathbf{A}_2\right)\\
&= \det\left[diag\left(\tilde{\mathbf{D}}_{L/2+1} + \mathbf{A}_{|S_{L/2+1}|}(K_1), \tilde{\mathbf{D}}_{L/2+2} + \mathbf{A}_{|S_{L/2+2}|}(K_2), \ldots, \tilde{\mathbf{D}}_L + \mathbf{A}_{|S_L|}(K_{L/2})\right)\right]\\
&= \prod_{i=1}^{L/2} \det\left(\tilde{\mathbf{D}}_{L/2+i} + \mathbf{A}_{|S_{L/2+i}|}(K_i)\right)
\end{aligned}$$

by lemma 1,

$$\begin{aligned}
\det\left(\tilde{\mathbf{D}}_{L/2+i} + \mathbf{A}_{|S_{L/2+i}|}(K_i)\right) &= K_i \prod_{j\in S_{L/2+i}} \frac{N_o}{p_j}\left[\frac{1}{K_i} + \sum_{j\in S_{L/2+i}} \frac{p_j}{N_o}\right]\\
&= \prod_{j\in S_{L/2+i}} \frac{N_o}{p_j}\left[1 + \frac{K_i P_{(L/2+i)}}{N_o}\right]
\end{aligned}$$

Therefore,

$$\det\left(\mathbf{A}_3 - \mathbf{A}_2{}^T\mathbf{A}_1{}^{-1}\mathbf{A}_2\right) = \prod_{i=1}^{L/2}\left\{\prod_{j\in S_{L/2+i}} \frac{N_o}{p_j}\left[1 + \frac{K_i P_{(L/2+i)}}{N_o}\right]\right\}$$

$$\begin{aligned}
\det(\mathbf{A}_1) &= \prod_{i=1}^{L/2} \det(\mathbf{D}_i)\\
&= \prod_{i=1}^{L/2}\left[\prod_{j\in S_i} \frac{N_o}{p_j}\left[1 + \frac{LP_{(i)}}{N_o}\right]\right]
\end{aligned}$$

$$\begin{aligned}
\det\left(\mathbf{W}^{-1} + \mathbf{S}^T\mathbf{S}\right) &= \prod_{i=1}^{L/2}\left[\prod_{j\in S_i} \frac{N_o}{p_j}\left[1 + \frac{LP_{(i)}}{N_o}\right]\right] \cdot \prod_{i=1}^{L/2}\left\{\prod_{j\in S_{L/2+i}} \frac{N_o}{p_j}\left[1 + \frac{K_i P_{(L/2+i)}}{N_o}\right]\right\}\\
&= \left(\prod_{j=1}^{K} \frac{N_o}{p_j}\right)\left\{\prod_{i=1}^{L/2}\left[1 + \frac{K_i P_{(L/2+i)}}{N_o}\right]\left[1 + \frac{LP_{(i)}}{N_o}\right]\right\}
\end{aligned}$$

$$\begin{aligned}
C_{SUM}(BAPS) &= \frac{1}{L}\log\left\{\left(\prod_{j=1}^{K} \frac{p_j}{N_o}\right)\left(\prod_{j=1}^{K} \frac{N_o}{p_j}\right)\left[\prod_{i=1}^{L/2}\left(1 + \frac{K_i P_{(L/2+i)}}{N_o}\right)\left(1 + \frac{LP_{(i)}}{N_o}\right)\right]\right\}\\
&= \frac{1}{L}\log\left\{\prod_{i=1}^{L/2}\left(1 + \frac{K_i P_{(L/2+i)}}{N_o}\right)\left(1 + \frac{LP_{(i)}}{N_o}\right)\right\}\\
&= \frac{1}{L}\log\left\{\prod_{i=1}^{L/2}\left(1 + \frac{LP_{(L/2+i)}}{N_o} - \frac{R^2 P_{(i)} P_{(L/2+i)}}{N_o\left(N_o + LP_{(i)}\right)}\right)\left(1 + \frac{LP_{(i)}}{N_o}\right)\right\}
\end{aligned}$$

2.3 Asymptotic Upper Bound of the sum capacity for selected sequence sets

In Section 2.2, we have evaluated the sum capacity formulas for Walsh Code, m-sequence and Binary Almost Perfect Sequence. In this section, we will find out the asymptotic upper bound of these formulas by letting the number of user K goes to infinity while maintaining the total power P_{tot} as a constant. We will first show the upper bound of the sum capacities by considering the continuous case and then prove that the asymptotic upper bound is actually equal to that upper bound.

2.3.1 Upper bound of the sum capacity for selected sequence sets

Walsh Code

By proposition 1, the sum capacity of the S-CDMA system employing Walsh Code is:

$$C_{SUM}(Walsh) = \frac{1}{L}\log\left\{\prod_{j=1}^{L}\left[1+\frac{LP_{(j)}}{N_o}\right]\right\}$$

where $P_{(j)}$ is the sum of power of all users assigned the sequence $\underline{\mathbf{s}}_j$.

In order to maximize the sum capacity, we need to seek the optimal sequences assignment based on the power of all users, such that the product

$$\prod_{j=1}^{L}\left[1+\frac{LP_{(j)}}{N_o}\right]$$

is maximized. Simply say, we have to partition all the users into L groups, such that the sum of power of each group can maximize the above product.

However, this problem is a typical problem in discrete optimization and does not have an explicit solution. The only way to seek the optimal solution is by applying some efficient algorithms.

Instead of solving the discrete optimization problem to find the maximum value, we consider the continuous case to find the upper bound. We assume all the $P_{(j)}$'s are real numbers while the total power P_{tot} maintains a fixed finite value, i.e. $P_{(1)} + P_{(2)} + \cdots + P_{(L)} = P_{tot}$. Otherwise, the sum capacity can be infinite.

We find the maximum value of the sum capacity for continuous case by programming. The maximum value is also an upper bound for discrete case.

Proposition 4:

$$\begin{aligned} &\text{maximize} && \frac{1}{L}\log\left\{\prod_{j=1}^{L}\left[1+\frac{LP_{(j)}}{N_o}\right]\right\} \\ &\text{subject to} && P_{(1)} + P_{(2)} + \cdots + P_{(L)} = P_{tot} \\ & && P_{(i)} \geq 0,\ P_{(i)} \in \Re \qquad \forall i = 1,2,\cdots,L \end{aligned}$$

The solution of above optimization problem is $P_{(j)} = \frac{P_{tot}}{L}\quad \forall j = 1,2,\cdots,L$ and the maximum value is $\log\left(1+\frac{P_{tot}}{N_o}\right)$.

The proof is left in Appendix.

m-sequence

By proposition 2, the sum capacity of the S-CDMA system employing m-sequence is:

$$C_{SUM}(m-seq) = \frac{1}{L}\log\left\{\prod_{j=1}^{L}\left[1+\frac{(L+1)P_{(j)}}{N_o}\right]\cdot\left(1-\sum_{i=1}^{L}\frac{P_{(i)}}{N_o+(L+1)P_{(i)}}\right)\right\}$$

where $P_{(j)}$ is the sum of power of all users assigned the sequence $\underline{\mathbf{s}}_j$.

Similar to the Walsh Code, we would like to find the upper bound by considering the continuous case.

Proposition 5:

$$\begin{aligned} &\text{maximize} && \frac{1}{L}\log\left\{\prod_{j=1}^{L}\left[1+\frac{(L+1)P_{(j)}}{N_o}\right]\cdot\left(1-\sum_{i=1}^{L}\frac{P_{(i)}}{N_o+(L+1)P_{(i)}}\right)\right\} \\ &\text{subject to} && P_{(1)}+P_{(2)}+\cdots+P_{(L)}=P_{tot} \\ &&& P_{(i)}\geq 0,\ P_{(i)}\in\Re \qquad \forall i=1,2,\cdots,L \end{aligned}$$

The solution of above optimization problem is $P_{(j)}=\frac{P_{tot}}{L}\quad \forall j=1,2,\cdots,L$ and the maximum value is $\log\left(1+\frac{(L+1)P_{tot}}{LN_o}\right)+\frac{1}{L}\log\left[\frac{LN_o+P_{tot}}{LN_o+(L+1)P_{tot}}\right]$.

The proof is left in Appendix.

2.3.2 Asymptotic Upper Bound

Walsh Code

Now we focus on the asymptotic case that the upper bound of the users' power approaches zero while the total power P_{tot} maintains a fixed finite value.

$$C_{SUM}(Walsh)=\frac{1}{L}\log\left\{\prod_{j=1}^{L}\left[1+\frac{LP_{(j)}}{N_o}\right]\right\}$$

One way to partition the user is as following:

1. Randomly partition the user into L groups;
2. At each time, pick up a user and move to another group only if the product or the sum capacity increases.
3. End until no more such user.

Finally we would reach a so-called *sub-optimal* state that changing any one of the user's sequence would not increase the sum capacity.

For example, $L = 4$, $K = 9$, $N_o = 1$. We can distribute the users as below:

Sequence	$\underline{\mathbf{s}}_1$	$\underline{\mathbf{s}}_2$	$\underline{\mathbf{s}}_3$	$\underline{\mathbf{s}}_4$
User Power	$p_1(10)$	$p_2(2)$	$p_3(6)$	$p_4(15)$
	$p_8(9)$	$p_5(7)$	$p_6(8)$	
		$p_7(12)$	$p_9(3)$	
Sum of power	$p_1 + p_8(19)$	$p_2 + p_5 + p_7(21)$	$p_3 + p_6 + p_9(17)$	$p_4(15)$

In this example, $P_{tot} = 72$, $P_{tot} / L = 18$. From the above distribution, the product $\prod_{j=1}^{L}\left[1+\frac{LP_{(j)}}{N_o}\right] = (1+4\cdot 19)(1+4\cdot 21)(1+4\cdot 17)(1+4\cdot 15) = 27547905$. However, if we change user 2's signature sequence from $\underline{\mathbf{s}}_2$ to $\underline{\mathbf{s}}_4$, as below :

Sequence	$\underline{\mathbf{s}}_1$	$\underline{\mathbf{s}}_2$	$\underline{\mathbf{s}}_3$	$\underline{\mathbf{s}}_4$
User Power	$p_1(10)$		$p_3(6)$	$p_4(15)$
	$p_8(9)$	$p_5(7)$	$p_6(8)$	$p_2(2)$
		$p_7(12)$	$p_9(3)$	
Sum of power	$p_1 + p_8(19)$	$p_5 + p_7(19)$	$p_3 + p_6 + p_9(17)$	$p_2 + p_4(17)$

Now, $\prod_{j=1}^{L}\left[1+\frac{LP_{(j)}}{N_o}\right] = (1+4\cdot 19)(1+4\cdot 19)(1+4\cdot 17)(1+4\cdot 17) = 28227969$.

There is an increase by changing user 2's signature sequence. However, we cannot increase the sum capacity any more by changing any one of the user's signature sequence.

At this state, it is not guaranteed that the product is maximum. We may swap, move more than one user in a time or even regroup all the users in order to increase the sum capacity. In the above example, if we distribute the user as following:

Sequence	$\underline{\mathbf{s}}_1$	$\underline{\mathbf{s}}_2$	$\underline{\mathbf{s}}_3$	$\underline{\mathbf{s}}_4$
User Power	$p_1(10)$	$p_8(9)$	$p_3(6)$	$p_4(15)$
	$p_6(8)$	$p_5(7)$	$p_7(12)$	$p_9(3)$
		$p_2(2)$		
Sum of power	$p_1 + p_6(18)$	$p_8 + p_5 + p_2(18)$	$p_3 + p_7(18)$	$p_4 + p_9(18)$

$$\prod_{j=1}^{L}\left[1+\frac{LP_{(j)}}{N_o}\right] = (1+4\cdot 18)(1+4\cdot 18)(1+4\cdot 18)(1+4\cdot 18) = 28398241$$ which actually is the maximum achievable value of the product. Therefore, we call the state that the sum capacity cannot be increased by changing one's signature sequence as *sub-optimal* state. Although the sum capacity at the sub-optimal state may not be maximum, there are some interesting properties.

Further let the upper bound of users' power be ε, then $p_i \le \varepsilon$, $\forall i = 1,2,\cdots,K$ and $\overline{P} = P_{tot} / L$

Lemma 3:

When a distribution of user is in the sub-optimal state, i.e. changing one's signature sequence cannot increase the sum capacity, then

$$\overline{P} - \varepsilon \le P_{(j)} \le \overline{P} + \varepsilon \quad \forall j = 1,2,\cdots,L$$

Proof of Lemma 3:

We prove $P_{(j)} \le \bar{P}+\varepsilon \quad \forall j=1,2,\cdots,L$ by contradiction first. Assume there exist a m such that $P_{(m)} > \bar{P}+\varepsilon$, then there must also exist a n such that $P_{(n)} < \bar{P}$, otherwise, $\sum_{j=1}^{L} P_{(j)} > L\bar{P}+\varepsilon > P_{tot}$ which is impossible. Without loss of generality, we assume $m=1$ and $n=2$. Now, we change the signature sequence of a user whose power is $p^{'}$ from the 1st sequence to the 2nd sequence. As a result, the sum of power of the 1st sequence becomes $\tilde{P}_{(1)} = P_{(1)} - p^{'}$ while the sum of power of the 2nd sequence becomes $\tilde{P}_{(2)} = P_{(2)} + p^{'}$. The difference between the new product and the original product is:

$$
\begin{aligned}
&\prod_{j=1}^{L}\left[1+\frac{L\tilde{P}_{(j)}}{N_o}\right]-\prod_{j=1}^{L}\left[1+\frac{LP_{(j)}}{N_o}\right] \\
&=\prod_{j=3}^{L}\left[1+\frac{LP_{(j)}}{N_o}\right]\left[\left(1+\frac{L\tilde{P}_{(1)}}{N_o}\right)\left(1+\frac{L\tilde{P}_{(2)}}{N_o}\right)-\left(1+\frac{LP_{(1)}}{N_o}\right)\left(1+\frac{LP_{(2)}}{N_o}\right)\right] \\
&=\prod_{j=3}^{L}\left[1+\frac{LP_{(j)}}{N_o}\right]\left[\frac{L}{N_o}\left(\tilde{P}_{(1)}+\tilde{P}_{(2)}-P_{(1)}-P_{(2)}\right)+\frac{L^2}{N_o^{\,2}}\left(\tilde{P}_{(1)}\tilde{P}_{(2)}-P_{(1)}P_{(2)}\right)\right]
\end{aligned}
$$

Substitute $\tilde{P}_{(1)} = P_{(1)} - p^{'}$ $\tilde{P}_{(2)} = P_{(2)} + p^{'}$,

$$
\begin{aligned}
&\prod_{j=3}^{L}\left[1+\frac{LP_{(j)}}{N_o}\right]\left[\frac{L}{N_o}\left(\tilde{P}_{(1)}+\tilde{P}_{(2)}-P_{(1)}-P_{(2)}\right)+\frac{L^2}{N_o^{\,2}}\left(\tilde{P}_{(1)}\tilde{P}_{(2)}-P_{(1)}P_{(2)}\right)\right] \\
&=\prod_{j=3}^{L}\left[1+\frac{LP_{(j)}}{N_o}\right]\left\{\frac{L}{N_o}\left(P_{(1)}-p^{'}+P_{(2)}+p^{'}-P_{(1)}-P_{(2)}\right)+\right. \\
&\qquad\qquad\left.\frac{L^2}{N_o^{\,2}}\left[\left(P_{(1)}-p^{'}\right)\left(P_{(2)}+p^{'}\right)-P_{(1)}P_{(2)}\right]\right\} \\
&=\prod_{j=3}^{L}\left[1+\frac{LP_{(j)}}{N_o}\right]\left[\frac{L^2}{N_o^{\,2}}\left(P_{(1)}P_{(2)}+p^{'}P_{(1)}-p^{'}P_{(2)}-p^{'}p^{'}-P_{(1)}P_{(2)}\right)\right] \\
&=\prod_{j=3}^{L}\left[1+\frac{LP_{(j)}}{N_o}\right]\left[\frac{L^2}{N_o^{\,2}}p^{'}\left(P_{(1)}-P_{(2)}-p^{'}\right)\right]
\end{aligned}
$$

Since

$$P_{(1)} > \overline{P} + \varepsilon$$
$$P_{(2)} < \overline{P}$$
$$p' < \varepsilon$$

We have $P_{(1)} - P_{(2)} - p > 0$.

Therefore the difference of the products:

$$\prod_{j=1}^{L}\left[1+\frac{L\tilde{P}_{(j)}}{N_o}\right]-\prod_{j=1}^{L}\left[1+\frac{LP_{(j)}}{N_o}\right]=\prod_{j=3}^{L}\left[1+\frac{LP_{(j)}}{N_o}\right]\left[\frac{L^2}{N_o^{\,2}}p^{'}\left(P_{(1)}-P_{(2)}-p^{'}\right)\right]$$
$$>0$$

This means that the new sum capacity after changing a user's signature sequence is larger than the original capacity. This contradicts to the assumption that the distribution of user is in sub-optimal state. Therefore, $P_{(j)} \le \overline{P} + \varepsilon \quad \forall j = 1,2,\cdots,L$.

The proof for $\overline{P} - \varepsilon \le P_{(j)} \quad \forall j = 1,2,\cdots,L$ is very similar and is omitted.

Note that $C_{SUM}\left(Walsh\right)$ is an increasing function of $P_{(j)}$, $\forall j = 1,2,\cdots,L$. By lemma 3, $C_{SUM}\left(Walsh\right)$ will always be larger than the worst case of $P_{(j)} = \overline{P} - \varepsilon \; \forall j = 1,2,\cdots,L$.

Therefore

$$C_{SUM}\left(Walsh\right) \ge \frac{1}{L}\log\left\{\prod_{j=1}^{L}\left[1+\frac{L\left(\overline{P}-\varepsilon\right)}{N_o}\right]\right\}$$

Let the upper bound of the users' power ε approaches zero, we have

$$\lim_{\varepsilon \to 0} C_{SUM}(Walsh) \geq \lim_{\varepsilon \to 0} \frac{1}{L} \log \left\{ \prod_{j=1}^{L} \left[1 + \frac{L(\bar{P} - \varepsilon)}{N_o} \right] \right\}$$

$$= \frac{1}{L} \log \left\{ \lim_{\varepsilon \to 0} \prod_{j=1}^{L} \left[1 + \frac{L(\bar{P} - \varepsilon)}{N_o} \right] \right\}$$

$$= \frac{1}{L} \log \left\{ \prod_{j=1}^{L} \left[1 + \frac{L(\bar{P})}{N_o} \right] \right\}$$

$$= \log \left(1 + \frac{P_{tot}}{N_o} \right)$$

By proposition 4, the sum capacity for Walsh Code is bounded above.

$$\log \left\{ \left(1 + \frac{P_{tot}}{N_o} \right) \right\} \geq C_{SUM}(Walsh)$$

So, we have

$$\log \left\{ \left(1 + \frac{P_{tot}}{N_o} \right) \right\} \geq C_{SUM}(Walsh) \geq \frac{1}{L} \log \left\{ \prod_{j=1}^{L} \left[1 + \frac{L(\bar{P} - \varepsilon)}{N_o} \right] \right\}$$

$$\log \left\{ \left(1 + \frac{P_{tot}}{N_o} \right) \right\} \geq \lim_{\varepsilon \to 0} C_{SUM}(Walsh) \geq \lim_{\varepsilon \to 0} \frac{1}{L} \log \left\{ \prod_{j=1}^{L} \left[1 + \frac{L(\bar{P} - \varepsilon)}{N_o} \right] \right\}$$

$$\log \left\{ \left(1 + \frac{P_{tot}}{N_o} \right) \right\} \geq \lim_{\varepsilon \to 0} C_{SUM}(Walsh) \geq \log \left\{ \left(1 + \frac{P_{tot}}{N_o} \right) \right\}$$

Therefore,

$$\lim_{\varepsilon \to 0} C_{SUM}(Walsh) = \log \left\{ \left(1 + \frac{P_{tot}}{N_o} \right) \right\}$$

m-sequence

Similar to Walsh Code case, we can always distribute the users randomly, then change one's signature sequence in order to increase the sum capacity and finally reach the sub-optimal state. We need 1 more lemma.

Lemma 4:

$$f(\underline{\mathbf{x}}) = \prod_{j=1}^{L}\left[1+\frac{(L+1)x_j}{N_o}\right]\cdot\left(1-\sum_{i=1}^{L}\frac{x_i}{N_o+(L+1)x_i}\right)$$ is an increasing function of $\underline{\mathbf{x}}$.

The proof is left in Appendix.

Lemma 5:

When a distribution of user is in the sub-optimal state, i.e. changing one's signature sequence cannot increase the sum capacity, then

$$\bar{P}-\varepsilon \le P_{(j)} \le \bar{P}+\varepsilon \quad \forall j=1,2,\cdots,L$$

Proof of Lemma 5:

Still, we prove $P_{(j)} \le \bar{P}+\varepsilon \quad \forall j=1,2,\cdots,L$ by contradiction first. Assume there exist a m such that $P_{(m)} > \bar{P}+\varepsilon$, then there must also exist a n such that $P_{(n)} < \bar{P}$, otherwise, $\sum_{j=1}^{L} P_{(j)} > L\bar{P}+\varepsilon > P_{tot}$ which is impossible. Without loss of generality, we assume $m=1$ and $n=2$. Now, we change the signature sequence of a user whose power is $p^{'}$ from the 1st sequence to the 2nd sequence. As a result, the sum of power of the 1st sequence becomes $\tilde{P}_{(1)} = P_{(1)} - p^{'}$ while the sum of power of the 2nd sequence becomes $\tilde{P}_{(2)} = P_{(2)} + p^{'}$. The difference between the new product and the original product is:

$$\prod_{j=1}^{L}\left[1+\frac{(L+1)\tilde{P}_{(j)}}{N_o}\right]\cdot\left(1-\sum_{i=1}^{L}\frac{\tilde{P}_{(i)}}{N_o+(L+1)\tilde{P}_{(i)}}\right)-$$
$$\prod_{j=1}^{L}\left[1+\frac{(L+1)P_{(j)}}{N_o}\right]\cdot\left(1-\sum_{i=1}^{L}\frac{P_{(i)}}{N_o+(L+1)P_{(i)}}\right)$$

$$= \prod_{j=3}^{L}\left[1+\frac{(L+1)P_{(j)}}{N_o}\right]\cdot\left\{\left[1+\frac{(L+1)\tilde{P}_{(1)}}{N_o}\right]\left[1+\frac{(L+1)\tilde{P}_{(2)}}{N_o}\right]\left(1-\sum_{i=1}^{L}\frac{\tilde{P}_{(i)}}{N_o+(L+1)\tilde{P}_{(i)}}\right)-\right.$$

$$\left.\left[1+\frac{(L+1)P_{(1)}}{N_o}\right]\left[1+\frac{(L+1)P_{(2)}}{N_o}\right]\left(1-\sum_{i=1}^{L}\frac{P_{(i)}}{N_o+(L+1)P_{(i)}}\right)\right\}$$

$$= \frac{1}{N_o^{\,2}}\prod_{j=3}^{L}\left[1+\frac{(L+1)P_{(j)}}{N_o}\right]\cdot\left\{\left[N_o+(L+1)\tilde{P}_{(1)}\right]\left[N_o+(L+1)\tilde{P}_{(2)}\right]\left(1-\sum_{i=1}^{L}\frac{\tilde{P}_{(i)}}{N_o+(L+1)\tilde{P}_{(i)}}\right)-\right.$$

$$\left.\left[N_o+(L+1)P_{(1)}\right]\left[N_o+(L+1)P_{(2)}\right]\left(1-\sum_{i=1}^{L}\frac{P_{(i)}}{N_o+(L+1)P_{(i)}}\right)\right\}$$

Let

$$D=\left[N_o+(L+1)\tilde{P}_{(1)}\right]\left[N_o+(L+1)\tilde{P}_{(2)}\right]\left(1-\sum_{i=1}^{L}\frac{\tilde{P}_{(i)}}{N_o+(L+1)\tilde{P}_{(i)}}\right)-$$

$$\left[N_o+(L+1)P_{(1)}\right]\left[N_o+(L+1)P_{(2)}\right]\left(1-\sum_{i=1}^{L}\frac{P_{(i)}}{N_o+(L+1)P_{(i)}}\right)$$

$$=\left\{\left[N_o+(L+1)\tilde{P}_{(1)}\right]\left[N_o+(L+1)\tilde{P}_{(2)}\right]\left(1-\sum_{i=3}^{L}\frac{P_{(i)}}{N_o+(L+1)P_{(i)}}\right)-\right.$$

$$\left.\tilde{P}_{(1)}\left[N_o+(L+1)\tilde{P}_{(2)}\right]-\tilde{P}_{(2)}\left[N_o+(L+1)\tilde{P}_{(1)}\right]\right\}-$$

$$\left\{\left[N_o+(L+1)P_{(1)}\right]\left[N_o+(L+1)P_{(2)}\right]\left(1-\sum_{i=3}^{L}\frac{P_{(i)}}{N_o+(L+1)P_{(i)}}\right)-\right.$$

$$\left.P_{(1)}\left[N_o+(L+1)P_{(2)}\right]-P_{(2)}\left[N_o+(L+1)P_{(1)}\right]\right\}$$

$$=\left(1-\sum_{i=3}^{L}\frac{P_{(i)}}{N_o+(L+1)P_{(i)}}\right)\left\{\left[N_o+(L+1)\tilde{P}_{(1)}\right]\left[N_o+(L+1)\tilde{P}_{(2)}\right]-\left[N_o+(L+1)P_{(1)}\right]\left[N_o+(L+1)P_{(2)}\right]\right\}+$$

$$P_{(1)}\left[N_o+(L+1)P_{(2)}\right]+P_{(2)}\left[N_o+(L+1)P_{(1)}\right]-\tilde{P}_{(1)}\left[N_o+(L+1)\tilde{P}_{(2)}\right]-\tilde{P}_{(2)}\left[N_o+(L+1)\tilde{P}_{(1)}\right]$$

Substitute $\tilde{P}_{(1)}=P_{(1)}-p'\ \tilde{P}_{(2)}=P_{(2)}+p'$,

$$D=\left(1-\sum_{i=3}^{L}\frac{P_{(i)}}{N_o+(L+1)P_{(i)}}\right)\left[(L+1)^2\,p'\left(P_{(1)}-P_{(2)}-p'\right)\right]+2(L+1)P_{(1)}P_{(2)}$$

$$-2(L+1)\left(P_{(1)}-p'\right)\left(P_{(2)}+p'\right)$$

Since

$$1-\sum_{i=3}^{L}\frac{P_{(i)}}{N_o+(L+1)P_{(i)}}>1-\sum_{i=3}^{L}\frac{P_{(i)}}{(L+1)P_{(i)}}$$
$$=1-\frac{L-2}{L+1}$$
$$=\frac{3}{L+1}$$

So,

$$\begin{aligned}D&>\frac{3}{L+1}\left[(L+1)^2p'\left(P_{(1)}-P_{(2)}-p'\right)\right]+2(L+1)P_{(1)}P_{(2)}-2(L+1)\left(P_{(1)}-p'\right)\left(P_{(2)}+p'\right)\\&=3(L+1)p'\left(P_{(1)}-P_{(2)}-p'\right)+2(L+1)P_{(1)}P_{(2)}-2(L+1)\left(P_{(1)}-p'\right)\left(P_{(2)}+p'\right)\\&=3(L+1)p'\left(P_{(1)}-P_{(2)}-p'\right)-2(L+1)p'\left(P_{(1)}-P_{(2)}-p'\right)\\&=(L+1)p'\left(P_{(1)}-P_{(2)}-p'\right)\\&>0\end{aligned}$$

This means that the new sum capacity after changing a user's signature sequence is larger than the original capacity. This contradicts to the assumption that the distribution of user is in sub-optimal state. Therefore, $P_{(j)}\le\overline{P}+\varepsilon\quad\forall j=1,2,\cdots,L$.

The proof for $\overline{P}-\varepsilon\le P_{(j)}\quad\forall j=1,2,\cdots,L$ is very similar and is omitted.

Note that by lemma 4, $C_{SUM}(m-seq)$ is an increasing function of $P_{(j)}$, $\forall j=1,2,\cdots,L$. By lemma 5, $C_{SUM}(m-seq)$ will always be larger than the worst case of $P_{(j)}=\overline{P}-\varepsilon\ \forall j=1,2,\cdots,L$.

Therefore,

$$C_{SUM}(m-seq)=\frac{1}{L}\log\left\{\prod_{j=1}^{L}\left[1+\frac{(L+1)(\overline{P}-\varepsilon)}{N_o}\right]\cdot\left(1-\sum_{i=1}^{L}\frac{(\overline{P}-\varepsilon)}{N_o+(L+1)(\overline{P}-\varepsilon)}\right)\right\}$$

Let the upper bound of the users' power ε approaches zero, we have

$$\lim_{\varepsilon \to 0} C_{SUM}\left(m-seq\right) \geq \lim_{\varepsilon \to 0} \frac{1}{L}\log\left\{\prod_{j=1}^{L}\left[1+\frac{(L+1)\left(\overline{P}-\varepsilon\right)}{N_o}\right]\cdot\left(1-\sum_{i=1}^{L}\frac{\left(\overline{P}-\varepsilon\right)}{N_o+(L+1)\left(\overline{P}-\varepsilon\right)}\right)\right\}$$

$$=\frac{1}{L}\log\left\{\prod_{j=1}^{L}\left[1+\frac{(L+1)\overline{P}}{N_o}\right]\cdot\left(1-\sum_{i=1}^{L}\frac{\overline{P}}{N_o+(L+1)\overline{P}}\right)\right\}$$

$$=\frac{1}{L}\log\left\{\prod_{j=1}^{L}\left[1+\frac{(L+1)\overline{P}}{N_o}\right]\cdot\left(\frac{N_o+\overline{P}}{N_o+(L+1)\overline{P}}\right)\right\}$$

$$=\log\left(1+\frac{(L+1)P_{tot}}{LN_o}\right)+\frac{1}{L}\log\left[\frac{LN_o+P_{tot}}{LN_o+(L+1)P_{tot}}\right]$$

By proposition 5, the sum capacity for m-sequence is bounded above.

$$\log\left(1+\frac{(L+1)P_{tot}}{LN_o}\right)+\frac{1}{L}\log\left[\frac{LN_o+P_{tot}}{LN_o+(L+1)P_{tot}}\right]\geq C_{SUM}\left(m-seq\right)$$

Therefore,

$$\lim_{\varepsilon \to 0} C_{SUM}\left(m-seq\right)=\log\left(1+\frac{(L+1)P_{tot}}{LN_o}\right)+\frac{1}{L}\log\left[\frac{LN_o+P_{tot}}{LN_o+(L+1)P_{tot}}\right]$$

2.4 Optimal dynamic code allocation scheme for ad-hoc S-CDMA System

In section 2.2, we have evaluated the sum capacity formulas for 3 different binary sequence sets: Walsh Code, m-sequence and Binary Almost Perfect Sequence. However, it is still an open problem for finding the binary sequence set that can maximize the sum capacity. The problem can be stated as below:

Assume there are K users and the length of signature sequence (processing gain) is L. Given the average-input-energy constraint for all users $\mathbf{W}$, let B be the set of all $L \times K$ binary matrices whose elements take the value +1 or -1 only. We would like to find the best **BINARY** sequence matrix $\mathbf{S}^*$ that maximizes the sum capacity. i.e.

$$C_{SUM}\left(\mathbf{S}^*\right) \geq C_{SUM}\left(\mathbf{S}\right) \qquad \forall \mathbf{S} \in B$$

However, if we consider an ad-hoc based system, i.e. users are joining the system one by one and a signature sequence is assigned to the user when he/she first enters the system, a code allocation scheme can be proved to be optimal at every stage.

2.4.1 Ad-hoc S-CDMA System

An ad-hoc S-CDMA System is a S-CDMA system that users enter the system one by one. At the time when a user joins, the base station would base on the strength of the power, assign a specific signature sequence dynamically.

The sum capacity formula derived in Section depends on the power and the signature sequence of all users. Thereby, adding a new user into the system would definitely change the value of sum capacity. We are seeking the optimal code allocation scheme that can maximize the sum capacity whenever a new user is added.

Assigning randomly generated signature sequence to new user is an effective scheme but not optimal. We now propose a simple code allocation scheme that can maximize the sum capacity of the system at every stage (whenever a new user is added).

2.4.2 Code allocation scheme

Assume the length of signature sequence (processing gain) is fixed and equals to L. The signature sequences that would be assigned to user in our scheme are mutually orthogonal (Walsh Code). The allocation of sequences depends on the power of the users.

Scheme:

Let L mutually orthogonal sequences be $\underline{\mathbf{o}}_1, \underline{\mathbf{o}}_2, \cdots, \underline{\mathbf{o}}_L$. We assign the first L users $\underline{\mathbf{o}}_1$ to $\underline{\mathbf{o}}_L$ respectively. Let $P_{(i)}$ be the sum of power of all users assigned $\underline{\mathbf{o}}_i$. Thus, up to now, $P_{(1)} = p_1, P_{(2)} = p_2, \ldots, P_{(L)} = p_L$. For the $(L+1)$-th user onward, we assign $\underline{\mathbf{o}}_j$ to the new user such that $P_{(j)}$ is the minimum among $P_{(1)}$ to $P_{(L)}$.

For example, $L=4$, $p_1=10$, $p_2=2$, $p_3=5$, $p_4=15$, $p_5=7$, $p_6=8$, $p_7=12$, $p_8=9$, $p_9=4$, $p_{10}=4$.

When the first 4 users enter the system, assign $\underline{\mathbf{o}}_1, \underline{\mathbf{o}}_2, \underline{\mathbf{o}}_3, \underline{\mathbf{o}}_4$ to them respectively, so

Sequence	$\underline{\mathbf{o}}_1$	$\underline{\mathbf{o}}_2$	$\underline{\mathbf{o}}_3$	$\underline{\mathbf{o}}_4$
User Power	$p_1(10)$	$p_2(2)$	$p_3(5)$	$p_4(15)$
Sum of power	$p_1(10)$	$p_2(2)$	$p_3(5)$	$p_4(15)$

For the fifth user, since the sum of power of sequence 2 is the smallest, the fifth user would be assigned sequence 2 as signature sequence.

Sequence	$\underline{\mathbf{o}}_1$	$\underline{\mathbf{o}}_2$	$\underline{\mathbf{o}}_3$	$\underline{\mathbf{o}}_4$
User Power	$p_1(10)$	$p_2(2)$	$p_3(5)$	$p_4(15)$
		$p_5(7)$		
Sum of power	$p_1(10)$	$p_2 + p_5(9)$	$p_3(5)$	$p_4(15)$

For the sixth user, the sum of power of sequence 3 is the smallest. Therefore, the sixth user would be assigned sequence 3 as the signature sequence.

Sequence	$\underline{\mathbf{o}}_1$	$\underline{\mathbf{o}}_2$	$\underline{\mathbf{o}}_3$	$\underline{\mathbf{o}}_4$
User Power	$p_1(10)$	$p_2(2)$	$p_3(5)$	$p_4(15)$
		$p_5(7)$	$p_6(8)$	
Sum of power	$p_1(10)$	$p_2 + p_5(9)$	$p_3 + p_6(13)$	$p_4(15)$

For the seventh user:

Sequence	$\underline{\mathbf{o}}_1$	$\underline{\mathbf{o}}_2$	$\underline{\mathbf{o}}_3$	$\underline{\mathbf{o}}_4$
User Power	$p_1(10)$	$p_2(2)$	$p_3(5)$	$p_4(15)$
		$p_5(7)$	$p_6(8)$	
		$p_7(12)$		
Sum of power	$p_1(10)$	$p_2+p_5+p_7(21)$	$p_3+p_6(13)$	$p_4(15)$

For the eighth user:

Sequence	$\underline{\mathbf{o}}_1$	$\underline{\mathbf{o}}_2$	$\underline{\mathbf{o}}_3$	$\underline{\mathbf{o}}_4$
User Power	$p_1(10)$	$p_2(2)$	$p_3(5)$	$p_4(15)$
	$p_8(9)$	$p_5(7)$	$p_6(8)$	
		$p_7(12)$		
Sum of power	$p_1+p_8(19)$	$p_2+p_5+p_7(21)$	$p_3+p_6(13)$	$p_4(15)$

Ninth user:

Sequence	$\underline{\mathbf{o}}_1$	$\underline{\mathbf{o}}_2$	$\underline{\mathbf{o}}_3$	$\underline{\mathbf{o}}_4$
User Power	$p_1(10)$	$p_2(2)$	$p_3(5)$	$p_4(15)$
	$p_8(9)$	$p_5(7)$	$p_6(8)$	
		$p_7(12)$	$p_9(4)$	
Sum of power	$p_1+p_8(19)$	$p_2+p_5+p_7(21)$	$p_3+p_6+p_9(17)$	$p_4(15)$

Tenth user:

Sequence	$\underline{\mathbf{o}}_1$	$\underline{\mathbf{o}}_2$	$\underline{\mathbf{o}}_3$	$\underline{\mathbf{o}}_4$
User Power	$p_1(10)$	$p_2(2)$	$p_3(5)$	$p_4(15)$
	$p_8(9)$	$p_5(7)$	$p_6(8)$	$p_{10}(4)$
		$p_7(12)$	$p_9(4)$	
Sum of power	$p_1 + p_8(19)$	$p_2 + p_5 + p_7(21)$	$p_3 + p_6 + p_9(17)$	$p_4 + p_{10}(19)$

In the following section, we give out the proof of the optimality of the above scheme.

2.4.3 Proof of Optimality

According to our scheme, it is obvious that the sum capacity is maximized for the first L users since each of them is assigned a mutually orthogonal sequence. For the $(L+1)$ user onward, we will show that our scheme is optimal by Mathematic Induction. We will prove that if there are already K users, $K>L$, each of them is assigned one of L mutually orthogonal sequence, when a new user is added to the system; the sum capacity is maximized by assigning the orthogonal sequence with smallest sum of power.

Proposition 6:

Consider an S-CDMA system with K users and power levels, $p_1, p_2, \cdots, p_K$. Moreover, each user is assigned one of L orthogonal sequences, $\underline{\mathbf{o}}_1, \underline{\mathbf{o}}_2, \cdots, \underline{\mathbf{o}}_L$, $K \geq L$. If one new user with power p_{K+1} is added to the system, and if no re-assignment is allowed for current users, then the maximum sum capacity is achieved by assigning the new user to use the spreading sequence $\underline{\mathbf{o}}_l$ that has the smallest user power sum,

$P_{(l)}$.

Proof of Proposition 6:

From equation(2.15),

$$C_{sum}(\mathbf{S}) = \frac{1}{L}\log\left[\left[\det\left(\mathbf{W}^{\frac{1}{2}}\right)\right]^2 \det\left(\mathbf{W}^{-1} + \mathbf{S}^T\mathbf{S}\right)\right]$$

where $\left[\det\left(\mathbf{W}^{\frac{1}{2}}\right)\right]^2 = \prod_{i=1}^{K}\frac{p_i}{N_o}$ and the term is independent of the signature sequence assignment to the users. Therefore, we only need to compare the value of the term $\det\left(\mathbf{W}^{-1} + \mathbf{S}^T\mathbf{S}\right)$.

Initially, there are K users using mutually orthogonal sequences, i.e. Walsh Code. From Section 2.1.1, we can write

$$\mathbf{W}^{-1} + \mathbf{S}^T\mathbf{S} = diag\left(\mathbf{D}_1, \mathbf{D}_2, \ldots, \mathbf{D}_L\right)$$

where

$$\mathbf{D}_j = \begin{bmatrix} L + \dfrac{N_o}{p.} & L & \ldots & L \\ L & L + \dfrac{N_o}{p.} & & L \\ \vdots & & \ddots & \vdots \\ L & L & \ldots & L + \dfrac{N_o}{p.} \end{bmatrix}$$

is a $|S_j| \times |S_j|$ square matrix whose diagonal elements contain the power of user $i, \forall i \in S_j$.

Let $\mathbf{U}_K = \mathbf{W}^{-1} + \mathbf{S}^T\mathbf{S}$. The subscript K means there are K users.

Without loss of generality, we assume that $P_{(1)} \geq P_{(2)} \geq P_{(3)} \geq \cdots \geq P_{(L)}$.

When one more user is added, assign to it the spreading sequence $\underline{\mathbf{o}}_L$ which has the smallest user power sum, $P_{(L)}$. The new matrix $\mathbf{W}^{-1}+\mathbf{S}^T\mathbf{S}$ for $K+1$ users becomes,

$$\mathbf{W}^{-1}+\mathbf{S}^T\mathbf{S}=\begin{bmatrix} \mathbf{D}_1 & & & & & & & & & 0 \\ & \mathbf{D}_2 & & & & & & & & 0 \\ & & & & & & & & & 0 \\ & & \mathbf{D}_3 & & & & & & & 0 \\ & & & & & & & & & 0 \\ & & & \ddots & & & & & & \vdots \\ & & & & & & & & & L \\ & & & & & & \mathbf{D}_L & & & \vdots \\ & & & & & & & & & L \\ 0 & 0 & 0 & 0 & 0 & \cdots & L & \cdots & L & L+\dfrac{N_o}{p_{K+1}} \end{bmatrix}$$

Denote it as $\hat{\mathbf{U}}_{K+1}$.

If we assign to the new user any other spreading sequence, the matrix $\mathbf{W}^{-1}+\mathbf{S}^T\mathbf{S}$ for $K+1$ users becomes:

$$\mathbf{W}^{-1}+\mathbf{S}^T\mathbf{S}=\det\begin{bmatrix} \mathbf{D}_1 & & & & & & & & & R_1 \\ & \mathbf{D}_2 & & & & & & & & R_2 \\ & & & & & & & & & R_3 \\ & & \mathbf{D}_3 & & & & & & & R_3 \\ & & & & & & & & & R_3 \\ & & & \ddots & & & & & & \vdots \\ & & & & & & & & & R_L \\ & & & & & & \mathbf{D}_L & & & R_L \\ & & & & & & & & & R_L \\ R_1 & R_2 & R_3 & R_3 & R_3 & \cdots & R_L & R_L & R_L & L+\dfrac{N_o}{p_{K+1}} \end{bmatrix}$$

where R_i is the correlation between the new spreading sequence and $\underline{\mathbf{o}}_i$ for i=1,2,…,L.

Denote it as $\tilde{\mathbf{U}}_{K+1}$

Since both $\det\left(\hat{\mathbf{U}}_{K+1}\right)$ and $\det\left(\tilde{\mathbf{U}}_{K+1}\right)$ have the following structure:

$$\det\begin{bmatrix} \mathbf{U}_K & \mathbf{A}_2{'} \\ \mathbf{A}_2 & \mathbf{A}_3 \end{bmatrix} = \det\left(\mathbf{U}_K\right)\det\left(\mathbf{A}_3 - \mathbf{A}_2\mathbf{U}_K{}^{-1}\mathbf{A}_2{'}\right)$$

Therefore,

$$\det\left(\hat{\mathbf{U}}_{K+1}\right) = \det\left(\mathbf{U}_K\right)\det\left(L + \frac{N_o}{p_{L+1}} - \begin{bmatrix}0 & 0 & \cdots & 0 & L & L & L\end{bmatrix}\mathbf{U}_K{}^{-1}\begin{bmatrix}0\\0\\\vdots\\0\\L\\L\\L\end{bmatrix}\right)$$

$$\det\left(\tilde{\mathbf{U}}_{K+1}\right) = \det\left(\mathbf{U}_K\right)\det\left(L + \frac{N_o}{p_{L+1}} - \begin{bmatrix}R_1 & R_2 & R_3 & \cdots & R_L & R_L & R_L\end{bmatrix}\mathbf{U}_K{}^{-1}\begin{bmatrix}R_1\\R_2\\R_3\\\vdots\\R_L\\R_L\\R_L\end{bmatrix}\right)$$

$$\det\left(\hat{\mathbf{U}}_{K+1}\right)-\det\left(\tilde{\mathbf{U}}_{K+1}\right)=\det\left(\mathbf{U}_K\right)\cdot$$

$$\left\{\begin{bmatrix}R_1 & R_2 & R_3 & \cdots & R_L & R_L & R_L\end{bmatrix}\mathbf{U}_K^{-1}\begin{bmatrix}R_1\\R_2\\R_3\\\vdots\\R_L\\R_L\\R_L\end{bmatrix}-\begin{bmatrix}0 & 0 & \cdots & 0 & L & L & L\end{bmatrix}\mathbf{U}_K^{-1}\begin{bmatrix}0\\0\\\vdots\\0\\L\\L\\L\end{bmatrix}\right\}$$

since

$$\mathbf{U}_K^{-1}=diag\left(\mathbf{D}_1^{-1},\mathbf{D}_2^{-1},\mathbf{D}_3^{-1},\ldots,\mathbf{D}_L^{-1}\right)$$

so

$$\det\left(\hat{\mathbf{U}}_{K+1}\right)-\det\left(\tilde{\mathbf{U}}_{K+1}\right)=\det\left(\mathbf{U}_K\right)\left\{\sum_{i=1}^{L}R_i^2 SUM\left(\mathbf{D}_i^{-1}\right)-L^2 SUM\left(\mathbf{D}_L^{-1}\right)\right\} \qquad (2.20)$$

By Lemma 2:

$$SUM\left(\mathbf{D}_k^{-1}\right)=\frac{\sum_{i\in O_k}\frac{p_i}{N_o}}{1+L\sum_{i\in O_k}\frac{p_i}{N_o}}=\frac{P_{(k)}}{N_o+LP_{(k)}}$$

Differentiate $SUM\left(\mathbf{D}_k^{-1}\right)$ with respect to $P_{(k)}$,

$$\begin{aligned}\frac{dSUM\left(D_k^{-1}\right)}{dP_{(k)}}&=\frac{N_o+LP_{(k)}-LP_{(k)}}{\left(N_o+LP_{(k)}\right)^2}\\&=\frac{N_o}{\left(N_o+LP_{(k)}\right)^2}\\&>0\end{aligned}$$

Thus $SUM\left(\mathbf{D}_k^{-1}\right)$ is a strictly increasing function of $P_{(k)}$.

As

$$P_{(1)}\geq P_{(2)}\geq P_{(3)}\geq\cdots\geq P_{(L)}$$

We have

$$SUM\left(\mathbf{D}_1^{-1}\right)\geq SUM\left(\mathbf{D}_2^{-1}\right)\geq SUM\left(\mathbf{D}_3^{-1}\right)\geq\cdots\geq SUM\left(\mathbf{D}_L^{-1}\right)$$

Apply it to equation(2.20),

$$
\begin{aligned}
\det\left(\hat{\mathbf{U}}_{K+1}\right)-\det\left(\tilde{\mathbf{U}}_{K+1}\right) &= \det\left(\mathbf{U}_{K}\right)\left\{\sum_{i=1}^{L} R_i^2 SUM\left(\mathbf{D}_i^{-1}\right)-L^2 SUM\left(\mathbf{D}_L^{-1}\right)\right\} \\
&\geq \det\left(\mathbf{U}_{K}\right)\left\{\sum_{i=1}^{L} R_i^2 SUM\left(\mathbf{D}_L^{-1}\right)-L^2 SUM\left(\mathbf{D}_L^{-1}\right)\right\} \\
&= \det\left(\mathbf{U}_{K}\right) SUM\left(\mathbf{D}_L^{-1}\right)\left\{\sum_{i=1}^{L} R_i^2 - L^2\right\}
\end{aligned}
$$

As

$$\sum_{i=1}^{L} R_i^{\,2} = L^2,$$

We have

$$\det\left(\hat{\mathbf{U}}_{K+1}\right)-\det\left(\tilde{\mathbf{U}}_{K+1}\right)\geq 0$$

Therefore, the sum capacity obtained by assigning the spreading sequence, $\underline{\mathbf{o}}_l$, that has the smallest user power sum, $P_{(l)}$, to the new user is larger than the sum capacity obtained by assigning any other spreading sequence to the new user.

Chapter 3

Simulation of code adaptation schemes for mobile CDMA System

3.1 Introduction of mobile CDMA System

In a spread spectrum, Code Division Multiple Access (CDMA) system using binary signaling, the radio signal received at the base station from the k^{th} mobile user (assuming no fading or multipath) is given by [14]

$$s_k(t-\tau_k) = \sqrt{2P_k}\, a_k(t-\tau_k) b_k(t-\tau_k)\cos(\omega_c t + \phi_k) \tag{3.1}$$

where $b_k(t)$ is the data sequence for user k, $a_k(t)$ is the spreading (or chip) sequence for user k, τ_k is the delay of user k, relative to some reference user 0, P_k is the received power of user k, and ϕ_k is the carrier phase offset of user k relative to a reference user 0. Since τ_k and ϕ_k are relative terms, we can define $\tau_0 = 0$, $\phi_0 = 0$.

Both $a_k(t)$ and $b_k(t)$ are binary sequences with values -1 and +1. The cyclical pseudo noise (PN) chip sequence $a_k(t)$ is of the form

$$a_k(t) = \sum_{j=-\infty}^{\infty} \sum_{i=0}^{L-1} a_{k,i} \prod\left(\frac{t-(i+jL)T_c}{T_c}\right) \qquad a_{k,i} \in \{-1,1\}$$

where L is the number of chips (*processing gain*) sent before the PN sequence repeats itself, T_c is the chip period, LT_c is the repetition period of the PN sequence, $\Pi(t)$ represents the unit pulse function, and i is an index to denote the particular chip within a PN chip cycle.

$$\Pi(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & \text{otherwise} \end{cases}$$

For the user data sequence, $b_k(t)$, T_b is the bit period. It is assumed that the bit period is an integer multiple of the chip period such that $T_b = LT_c$. In this case, the PN sequence is repeated for every bit period. The user data sequence $b_k(t)$ is given by

$$b_k(t) = \sum_{j=-\infty}^{\infty} b_{k,j} \Pi\left(\frac{t - jT_b}{T_b}\right) \qquad b_{k,j} \in \{-1, 1\}$$

In a mobile radio, spread spectrum, CDMA system, the signals from many users arrive at the input of the receiver. A correlation receiver is typically used to “filter” the desired user from all other users which share the same channel.

At the receiver, the signal available at the input to the matched-filter is given by

$$r(t) = \sum_{k=0}^{K-1} s_k(t - \tau_k) + n(t) \tag{3.2}$$

where $n(t)$ is additive Gaussian noise with two-sided power spectral density $N_o/2$. It is assumed that there is no multipath in the channel.

The received signal contains both the desired user and $(k-1)$ undesired users and is mixed down to baseband, multiplied by the PN sequence of the desired user (user 0, for example), and integrated over one bit period. Thus, assuming that the receiver is delay and phase synchronized with user 0, the decision statistic for user 0 is given by:

$$Z_0 = \int_{jT_b}^{(j+1)T_b} r(t) a_o(t) \cos(\omega_c t) dt$$

For convenience and simplicity of notation, the remainder of this analysis considers bit $0\,(j=0)$.

Substituting equations and into equation, the decision statistic of the receiver is found to be

$$Z_0 = \int_0^{T_b}\left[\left(\sum_{k=0}^{K-1}\sqrt{2P_k}\,a_k(t-\tau_k)b_k(t-\tau_k)\cos\left(\omega_c t+\phi_k\right)\right)+n(t)\right]a_o(t)\cos(\omega_c t)dt$$

which may be expressed as

$$Z_0 = I_0 + \eta + \zeta \tag{3.3}$$

where I_0 is the desired contribution to the decision statistic from the desired user $(k=0)$, ζ is the multiple access interference from all co-channel users (which may be in the same cell or in a different cell), and η is the thermal noise contribution.

The contribution from the desired user is given by

$$\begin{aligned}
I_0 &= \sqrt{2P_0}\int_0^{T_b} a_k^{\,2}(t)b_k^{\,2}(t)\cos^2\left(\omega_c t\right)dt \\
&= \sqrt{\frac{P_0}{2}}\int_0^{T_b}\left(\sum_{i=-\infty}^{\infty} b_{k,i}\,\Pi\left(\frac{t-iT_b}{T_b}\right)\right)\left(1+\cos\left(2\omega_c t\right)\right)dt \\
&= \sqrt{\frac{P_0}{2}}b_{k,0}\int_0^{T_b}\left(1+\cos\left(2\omega_c t\right)\right)dt \\
&\approx \sqrt{\frac{P_0}{2}}b_{k,0}T_b
\end{aligned}$$

The noise term, η, is given by:

$$\eta = \int_0^{T_b} n(t)a_0(t)\cos(\omega_c t)dt$$

The mean η is

$$\mu_\eta = E[\eta] = \int_0^{T_b} E[\eta]a_0(t)\cos(\omega_c t)dt = 0$$

The variance of η is

$$
\begin{aligned}
\sigma_\eta^{\,2} &= E\left[(\eta-\mu_\eta)^2\right] \\
&= E\left[\eta^2\right] \\
&= E\left[\int_0^{T_b}\int_0^{T_b} n(t)n(\lambda)a_0(t)a_0(\lambda)\cos(\omega_c t)\cos(\omega_c\lambda)\,dt\,d\lambda\right] \\
&= \int_0^{T_b}\int_0^{T_b} E\left[n(t)n(\lambda)\right]a_0(t)a_0(\lambda)\cos(\omega_c t)\cos(\omega_c\lambda)\,dt\,d\lambda \\
&= \int_0^{T_b}\int_0^{T_b} \frac{N_o}{2}\delta(t-\lambda)a_0(t)a_0(\lambda)\cos(\omega_c t)\cos(\omega_c\lambda)\,dt\,d\lambda \\
&= \frac{N_o}{2}\int_0^{T_b} a_0^{\,2}(t)\cos^2(\omega_c t)\,dt \\
&= \frac{N_o}{4}\int_0^{T_b}\left(1+\cos(2\omega_c t)\right)dt \\
&\approx \frac{N_o T_b}{4}
\end{aligned}
$$

The third component ζ , represents the contribution of multiple access interference to the decision statistic. ζ is the summation of $K-1$ terms, I_k ,

$$\zeta = \sum_{k=1}^{K-1} I_k \tag{3.4}$$

each of which is given by

$$I_k = \int_0^{T_b} \sqrt{2P_k}\, b_k(t-\tau_k)a_k(t-\tau_k)a_o(t)\cos(\omega_c t+\phi_k)\cos(\omega_c t)\,dt \tag{3.5}$$

The expected value of multiple access interference is difficult to evaluate because it requires a complete picture of the periodic cross-correlation between every pair of spreading sequences. A number of approximations were being proposed, for instance, Standard Gaussian Approximation (SGA) [14][20], Improved Gaussian Approximation (IGA) [14][18], and Simplified Expression of the Improved Gaussian Approximation (SEIGA) [19].

The Signal-to-Interference Ratio (SIR) at the receiver, which measures the ratio between the useful power and the amount of interference generated by all the other sources sharing the same resources, can be expressed as:

$$SIR = \frac{E\left[I_0^{\,2}\right]}{E\left[\zeta^2\right]+E\left[\eta^2\right]} \tag{3.6}$$

Mobile CDMA System refers to an asynchronous CDMA System with mobility. As the users move, the distances between the users and the base station are time-varying. The propagation time of signal will change accordingly. This results in time shifting of received signal at the base station. For example, in Figure 3.1, user 0 is stationary while user 1 is moving to the right with velocity v.

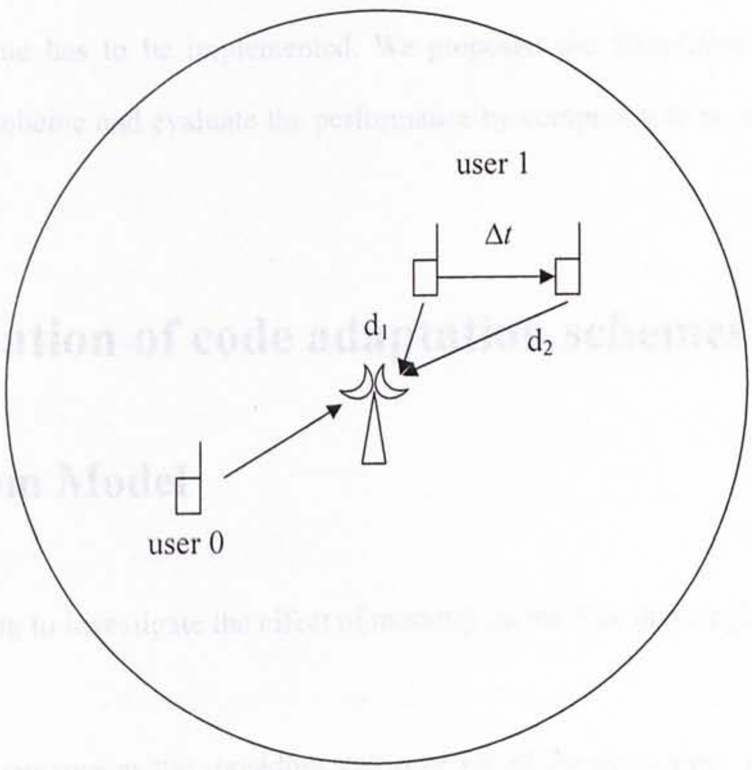

Figure 3.1: Effect of mobility on time delay

Assume the time delay of user 0, τ_0, is zero while the relative time delay of user 1 is τ_1 initially. After time Δt, the distance between user 1 and the base station increases from d_1 to d_2. The relative time delay of user 1 becomes $\tau_1 + (d_2 - d_1)/c$, where c is the speed of light. It can be observed that the time delay increased as the distance between the user and the base station increased. If $(d_2 - d_1)/c = T_c$, where T_c is the time duration of a chip, the received signal from user 1 has already shifted a chip when compared to the received signal from user 0. As a result, the multiple access interference caused by user 1, I_1, would change as user 1 keeps moving. If the received signal from user 1 shifted to a extent that the shifted spreading sequence of user 1, $a_1(t-\tau_1)$, is as same as the spreading sequence of user 0, $a_0(t)$, the multiple access interference I_1 would be very large. We call the coincidence of spreading sequences as "collision". The SIR of both users would degrade a lot for a period of time, until the spreading sequences differ again. In order to avoid collision, code adaptation scheme has to be implemented. We proposed the Simplified Maximum Collision Time scheme and evaluate the performance by comparing to no scheme and random scheme.

3.2 Simulation of code adaptation schemes

3.2.1 System Model

We are going to investigate the effect of mobility on the SIR in a single-cell CDMA system.

We use *m-sequence* as the spreading sequence for all the users since m-sequence has a number of advantages, such as easy to generate, nearly perfect auto and cross correlation. Thereby, each user is initially assigned the same m-sequence but with a different offset. We allow the users having a time-varying velocity with random direction. There are four types of users:

Type of user	Characteristics
Stationery	Stay at a fixed location all over the time
Roaming	Move around a fixed point with speed 3km/h-10km/h
Slow-moving	Move in a fixed direction with speed 10km/h-30km/h
Fast-moving	Move in a fixed direction with speed 20km/h-60km/h

Table 3.1: 4 types of user

For slow-moving and fast-moving users, both their direction and speed are changing randomly with time.

Since we concentrate on how the code adaptation schemes combat mobility, we further assume there is no fading and the power control is perfect, i.e. all received powers from all users are the same.

3.2.2 Descriptions of 3 schemes

In this section, we will look into how the schemes work and the characteristics of each scheme. For simplicity, we will consider there are only 4 users and the length of m-sequence (processing gain) is 15. There is exactly 1 user for each type of user and their motions are shown in Figure 3.2.

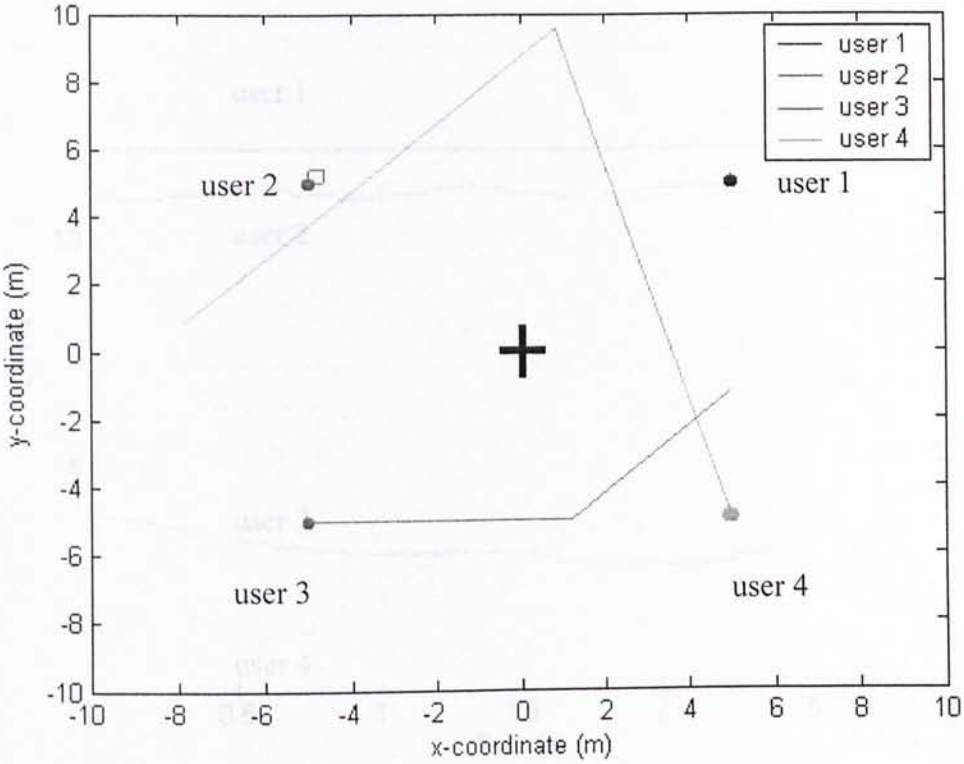

Figure 3.2: Trajectory of users

We can see that Type 1 user always stays at a point while Type 2 user is moving around a point. Type 3 and Type 4 users are moving with different speed and their directions and speeds change after some time.

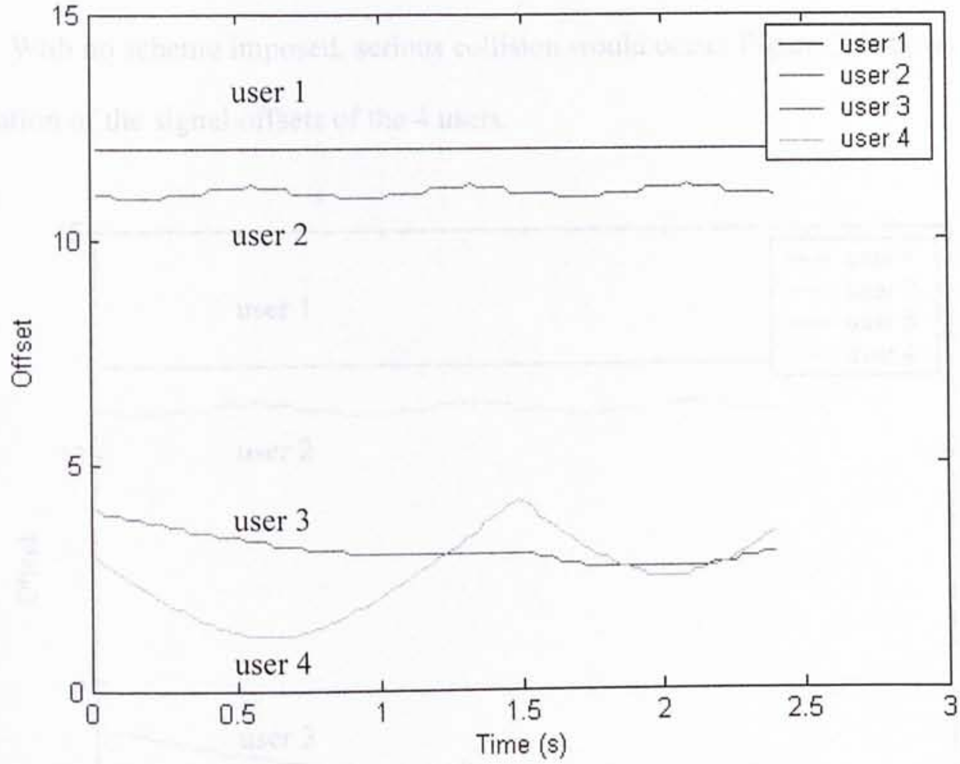

Figure 3.3: Variation of signal offsets at the receiver

As users move, the relative delays change. Figure 3.3 shows the variation between the offset of the received signals at the base station and time. Initially, user 1 is at offset 12, user 2 at offset 11, user 3 at offset 4 and user 4 at offset 3. As users move, the offset of the received signals at the base station vary. For example, user 4 first moves towards the base station from 0 to 0.6 second, so his relative offset shifts to be smaller; then he moves away from the base station from 0.6 to 1.5 second, as a result, his relative offset shifts to be larger.

In the next part, we will describe the detail of 3 schemes: no scheme, random scheme and Simplified Maximum Collision Time scheme. We will compare the performance of the 3 scheme base on the scenario shown in Figure 3.2.

1. No Scheme

With no scheme imposed, serious collision would occur. Figure 3.4 shows the variation of the signal offsets of the 4 users.

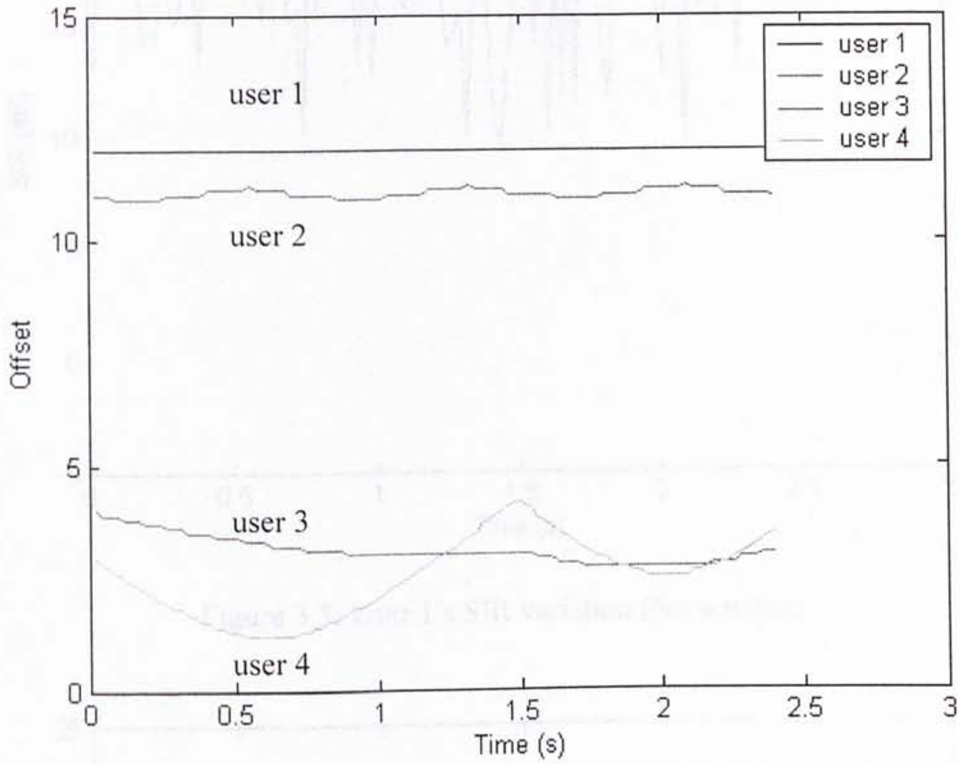

Figure 3.4: Variation of signal offsets (No scheme)

Notice that at time 1.2s, user 3's offset is very close to user 4's offset. As a result, the SIR of the 2 users involved in collision would degrade very much. Figure 3.5 to Figure 3.8 shows the SIR-time diagram for the 4 users. The degradation of SIR is very apparent.

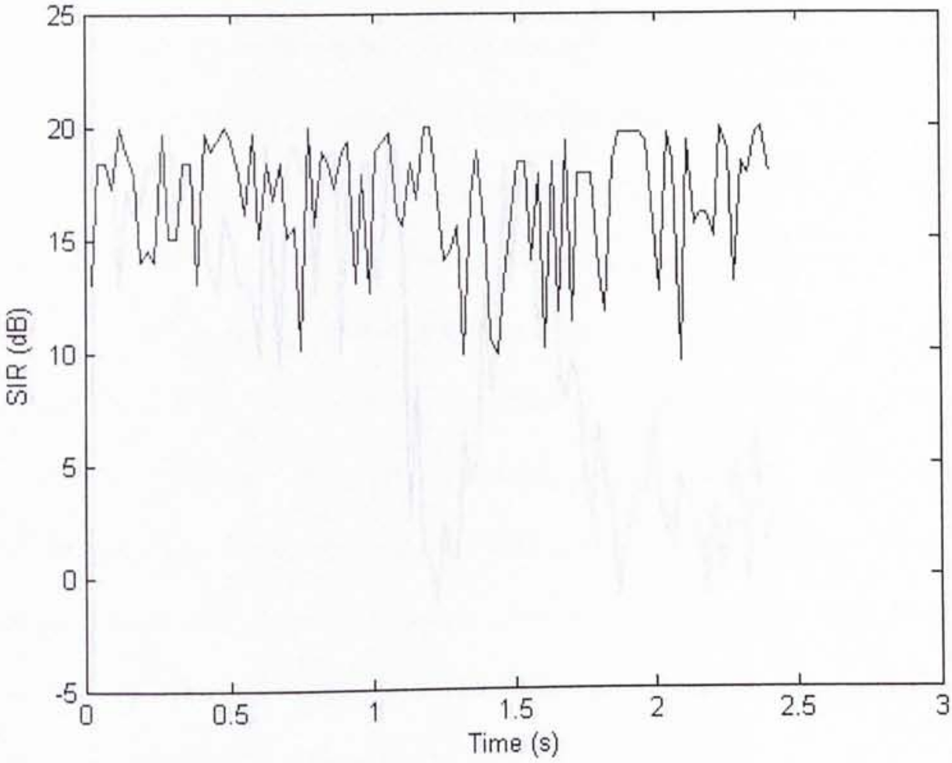

Figure 3.5: User 1's SIR variation (No scheme)

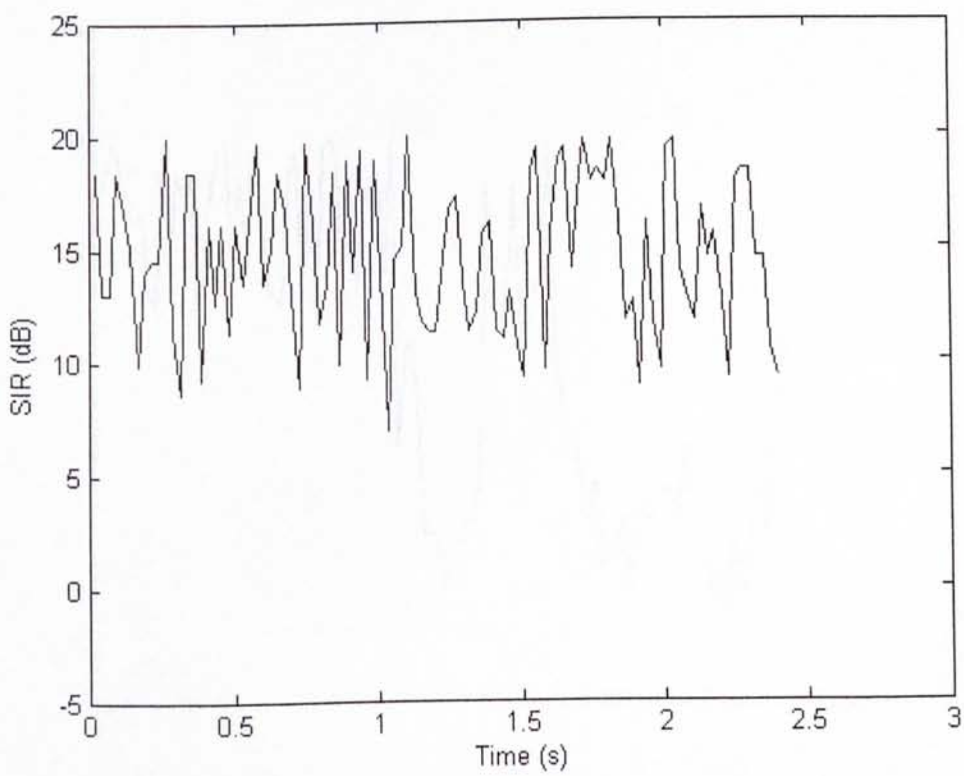

Figure 3.6: User 2's SIR variation (No scheme)

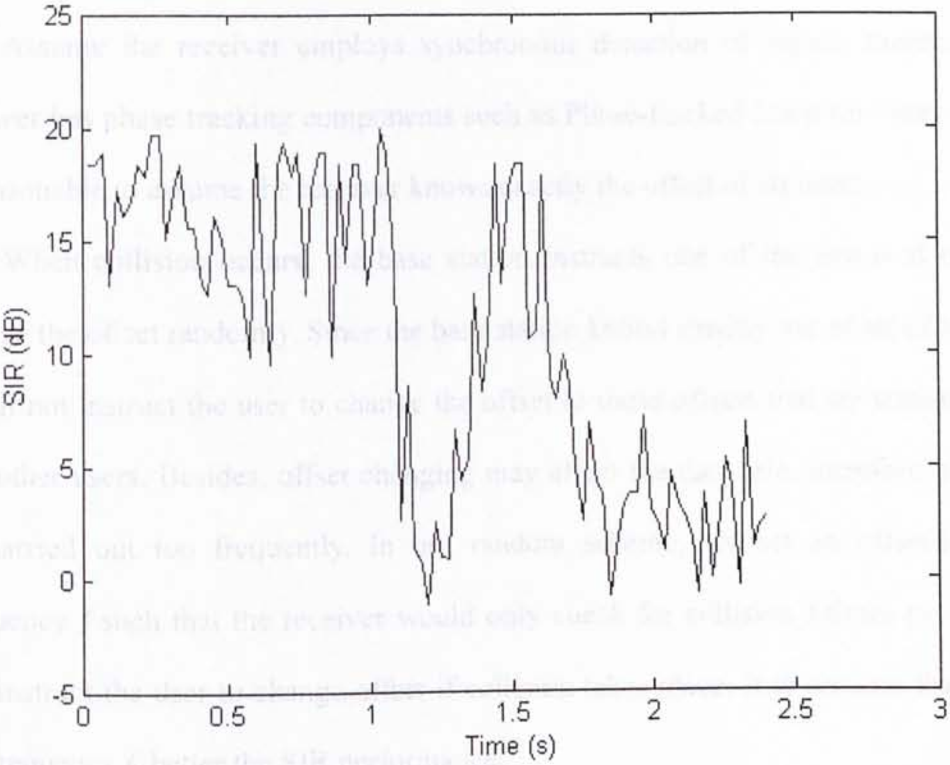

Figure 3.7: User 3's SIR variation (No scheme)

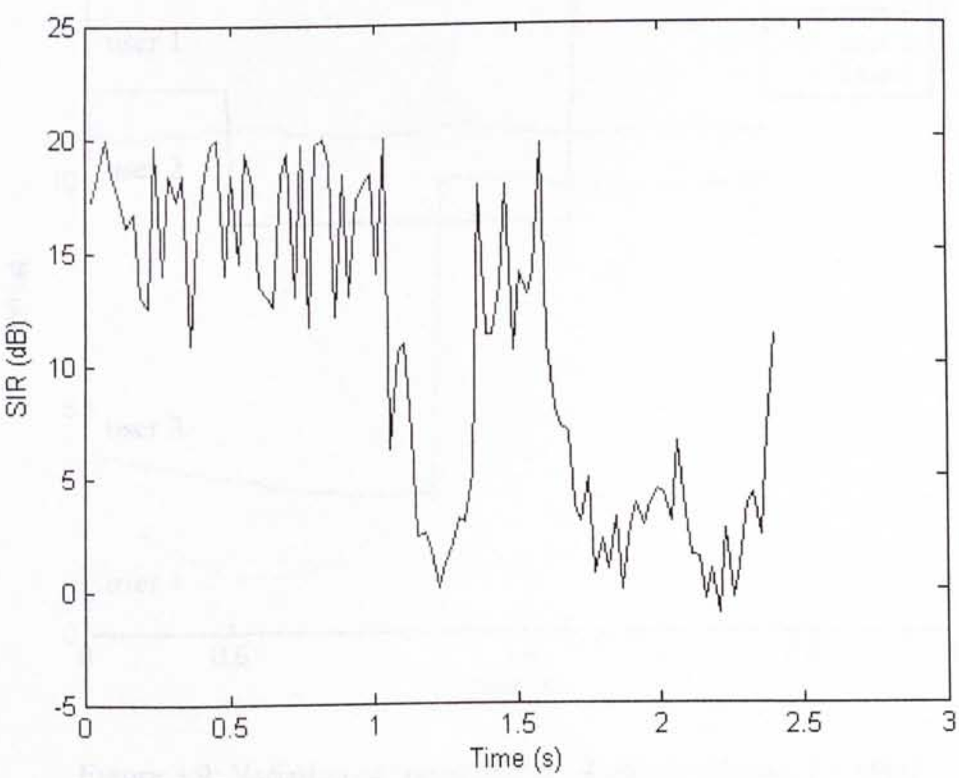

Figure 3.8: User 4's SIR variation (No scheme)

2. Random Scheme

Assume the receiver employs synchronous detection of signal, therefore, the receiver has phase tracking components such as Phase-Locked Loop for every user. It is reasonable to assume the receiver knows exactly the offset of all users.

When collision occurs, the base station instructs one of the involved users to change the offset randomly. Since the base station knows exactly the offset of all users, it will not instruct the user to change the offset to those offsets that are consumed by any other users. Besides, offset changing may affect the data rate, therefore it cannot be carried out too frequently. In our random scheme, we set an offset-jumping frequency f such that the receiver would only check for collision f times per second and instruct the user to change offset if collision takes place. It is obvious that larger the frequency f, better the SIR performance.

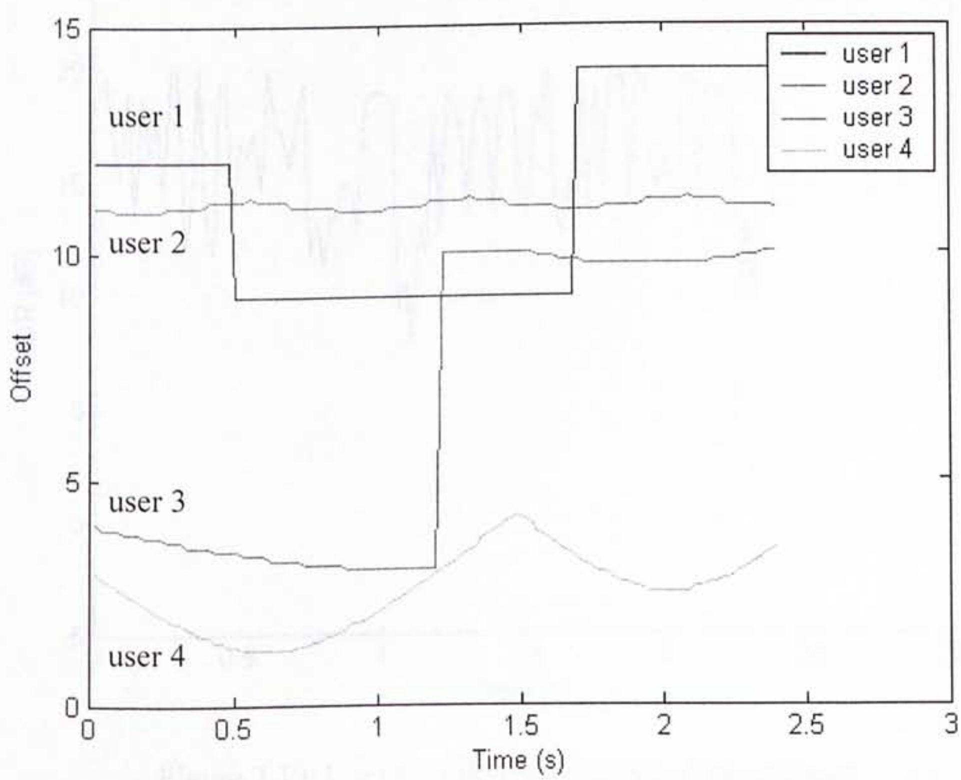

Figure 3.9: Variation of signal offsets (Random scheme, f = 4Hz)

As shown in Figure 3.9, at time 0.5s, the offset of user 1 is 12 while the offset of user 2 is 11. However, the offset of user 2 is still shifting to be larger, so collision occurs. The offset of user 1 would change randomly and it becomes 9. At time 1.25s, user 3 collides with user 4 and changes the offset again. This time, the offset becomes 10. When a collision takes place, it involves 2 users. The decision of which user changing the offset depends on policy. Changing offset may affect the transmission slightly.

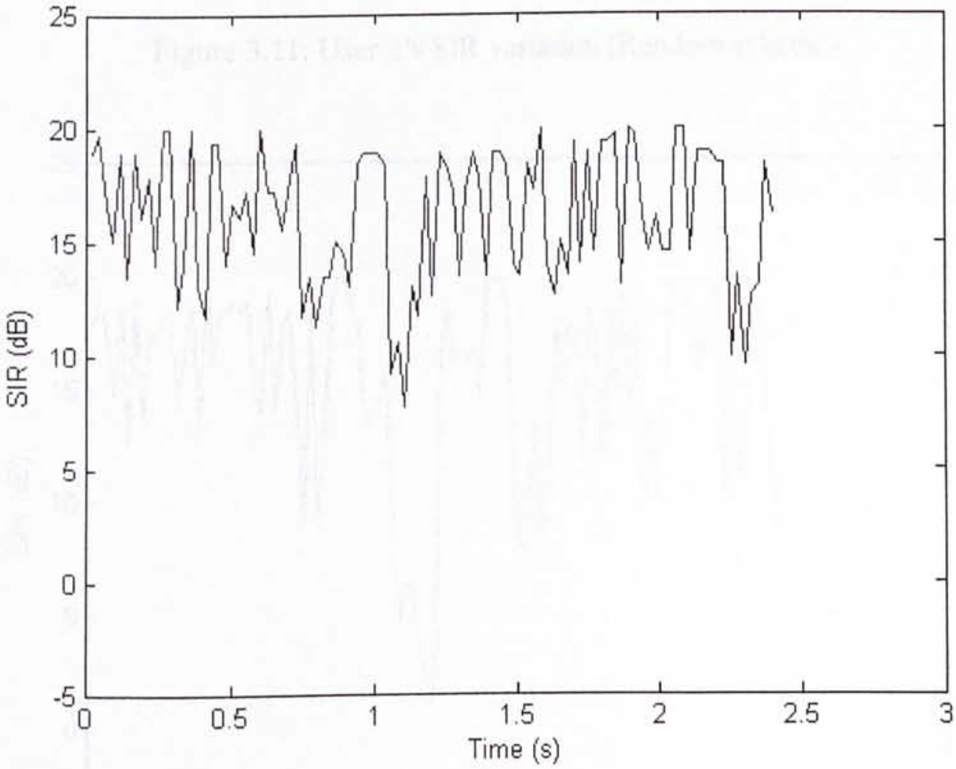

Figure 3.10: User 1's SIR variation (Random scheme)

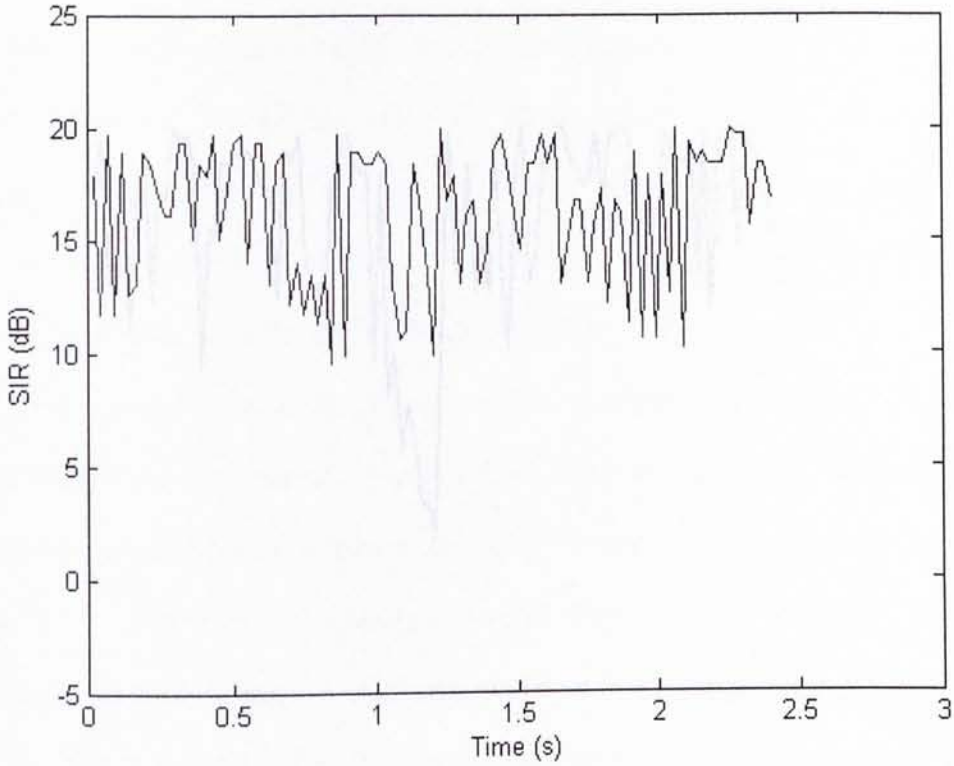

Figure 3.11: User 2's SIR variation (Random scheme)

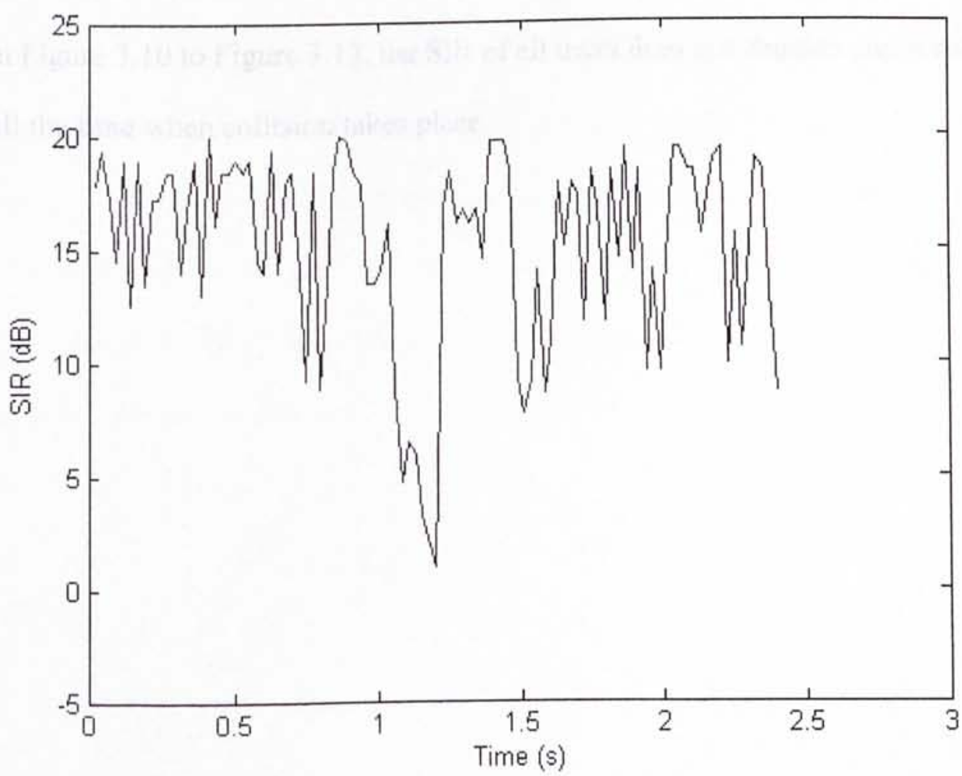

Figure 3.12: User 3's SIR variation (Random scheme)

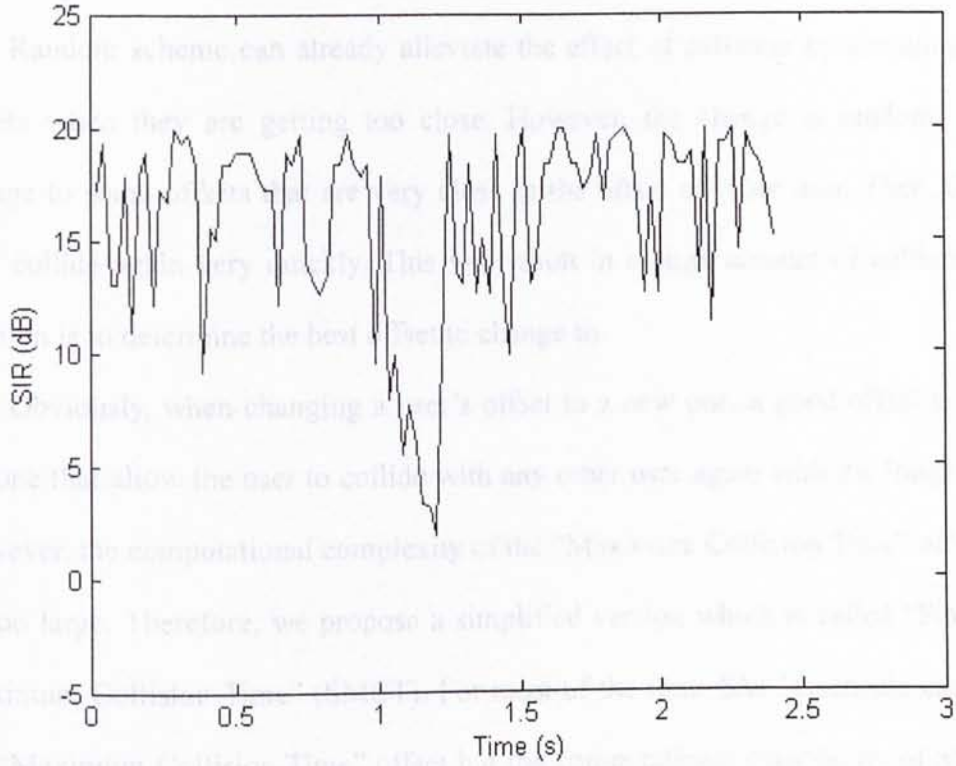

Figure 3.13: User 4's SIR variation (Random scheme)

From Figure 3.10 to Figure 3.13, the SIR of all users does not degrade significantly for all the time when collision takes place.

3. Simplified Maximum Collision Time

Random scheme can already alleviate the effect of collision by changing users' offsets when they are getting too close. However, the change is random. It may change to some offsets that are very close to the offset of other user. Then, the user may collide again very quickly. This will result in a large amount of collision. The solution is to determine the best offset to change to.

Obviously, when changing a user's offset to a new one, a good offset would be the one that allow the user to collide with any other user again with the longest time. However, the computational complexity of the "Maximum Collision Time" offset may be too large. Therefore, we propose a simplified version which is called "Simplified Maximum Collision Time" (SMCT). For most of the time, SMCT scheme can locate the "Maximum Collision Time" offset but the computational complexity of SCMT is much smaller.

We use the following diagram to represent the offsets. Suppose the length of m-sequence is 15 and there are 2 users. The first user is at offset 4, say, and the second user is at offset 10. Assume the first user is moving such that his offset is shifting to be larger (to the right) and the second user is moving also such that his offset is shifting to be smaller (to the left). Let the offset-shifting velocity be v_1 and v_2 for user 1 and 2 respectively. The offset-shifting velocity depends on the actually velocity of the users. We can represent the above situation by the following diagram:

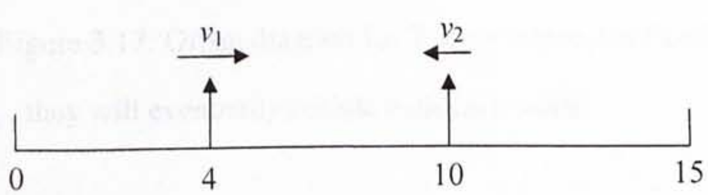

Figure 3.14: Offset diagram for 2 users (opposite direction)

After some time, they will collide with each other.

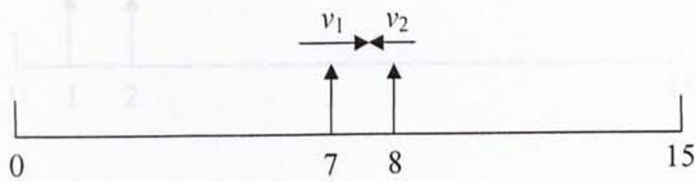

Figure 3.15: Offset diagram for 2 colliding users (opposite direction)

If user 1 needs to change his offset, since the direction of the offset-shifting velocities are different, so the best offset would be the offset just "behind" user 2.

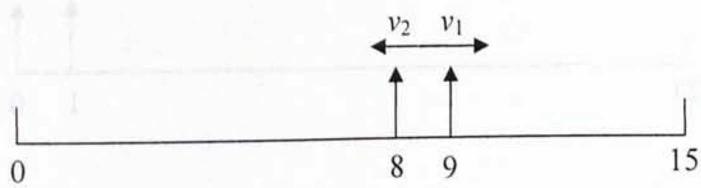

Figure 3.16: Offset diagram after jumping (opposite direction)

If both v_1 and v_2 are in the same direction,

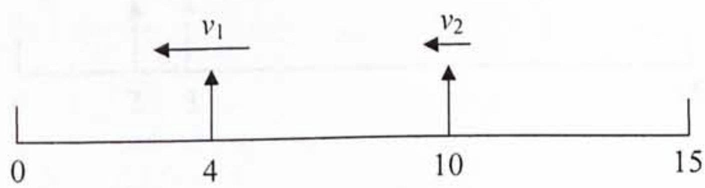

Figure 3.17: Offset diagram for 2 users (same direction)

Assume $v_1 > v_2$, they will eventually collide with each other.

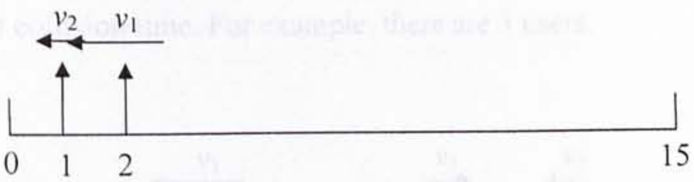

Figure 3.18: Offset diagram for colliding 2 users (same direction)

If offset 1 has to be changed, the best position is just "in front of" offset 2 since $v_1 > v_2$.

Figure 3.19: Offset diagram after jumping of user 1 (same direction)

If offset 2 has to be changed, the best position is just "behind" offset 1 since $v_1 > v_2$.

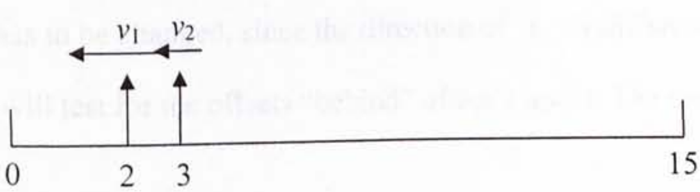

Figure 3.20: Offset diagram after jumping of user 2 (same direction)

Therefore, when the offset-shifting velocities are in the same direction, the position of new offset depends on the magnitude of the velocities.

From the above example, we find out that the "Maximum Collision Time" offset usually appear just "in front of" or "behind" other offsets. So, our scheme, "Simplified Maximum Collision Time" checks all these candidates and chooses the one with longest collision time. For example, there are 3 users,

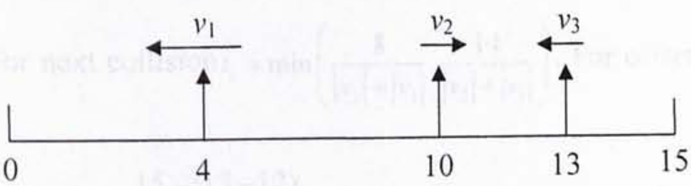

Figure 3.21: Offset diagram for 3 users

User 2 and 3 will collide shortly,

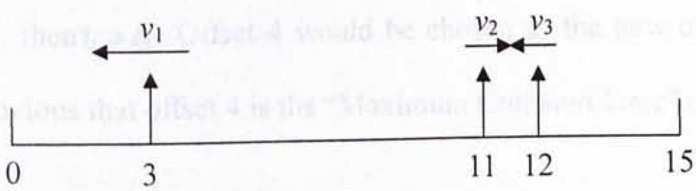

Figure 3.22: Offset diagram for collision

If offset 2 has to be changed, since the direction of v_2 is different from both v_1 and v_3, so we will test for the offsets "behind" offset 1 and 3. The candidates are offset 4 and 13.

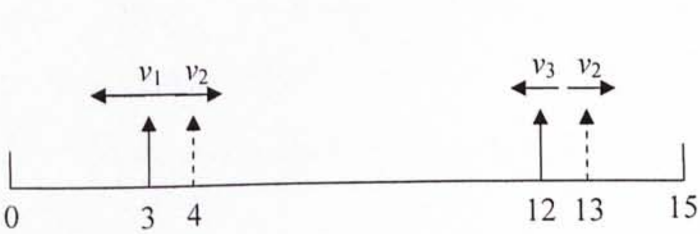

Figure 3.23: Candidates of "good" new offset

Then the time for next collision will be computed. For offset 4, the time to collide with user 3 is $\frac{12-4}{|v_2|+|v_3|}$, while the time to collide with user 1 is $\frac{15-(4-3)}{|v_2|+|v_1|}$. Thus, the time for next collision $t_1 = \min\left(\frac{8}{|v_2|+|v_3|}, \frac{14}{|v_2|+|v_1|}\right)$. For offset 13, the time to collide with user 3 is $\frac{15-(13-12)}{|v_2|+|v_3|}$, while the time to collide with user 1 is $\frac{15-(13-3)}{|v_2|+|v_1|}$. Thus, the time for next collision $t_2 = \min\left(\frac{14}{|v_2|+|v_3|}, \frac{5}{|v_2|+|v_1|}\right)$. Finally, by comparing t_1 and t_2, offset 4 is chosen if $t_1 > t_2$; otherwise, offset 13 is chosen. If assume $|v_1| > |v_3|$, then $t_1 > t_2$. Offset 4 would be chosen as the new offset for user 2. Actually, it is obvious that offset 4 is the "Maximum Collision Time" offset.

By applying the "Simplified Maximum Collision Time" scheme to the scenario shown in Figure 3.2, the variation of signal offsets is as below:

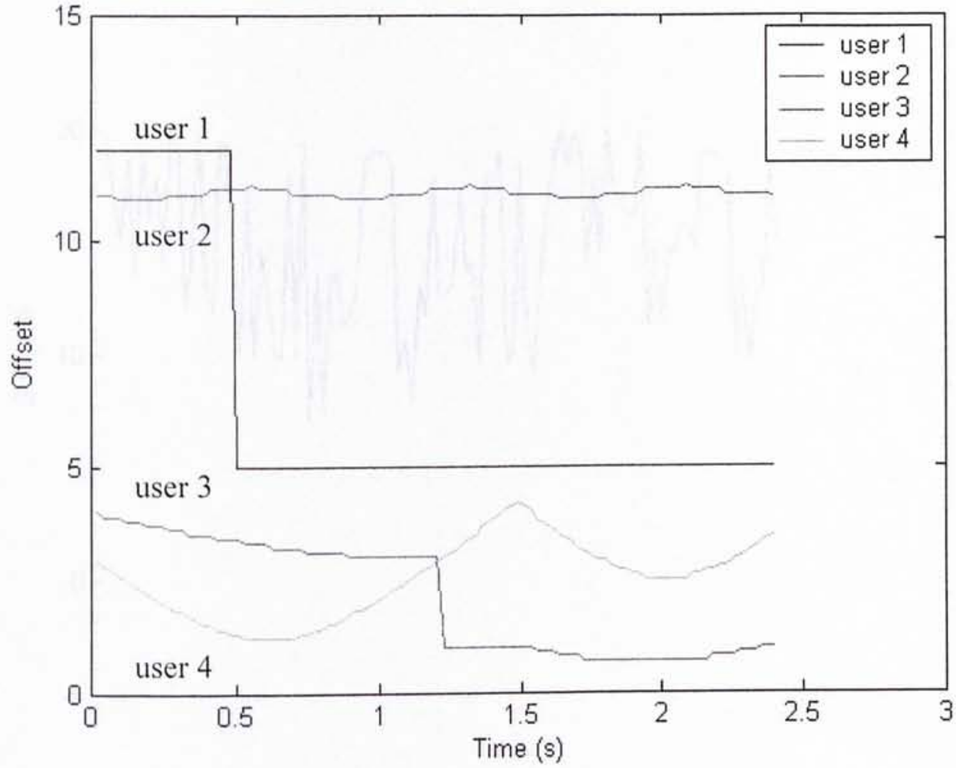

Figure 3.24: Variation of signal offsets (SMCT scheme, f = 4Hz)

As the receiver knows exactly the offsets of all users', it can keep track of the variation of the offsets. Thereby, the offset-shifting velocities can be estimated by regression. In the simulation, we use *quadratic regression* to accurately estimate the offset-shifting velocities.

Figure 3.25 to Figure 3.28 are the SIR-time diagram of the 4 users. The effect of Simplified Maximum Collision Time scheme is similar to random scheme. However, SMCT scheme has an advantage of having a smaller number of collisions and in turns, better performance. It will be demonstrated in the next section.

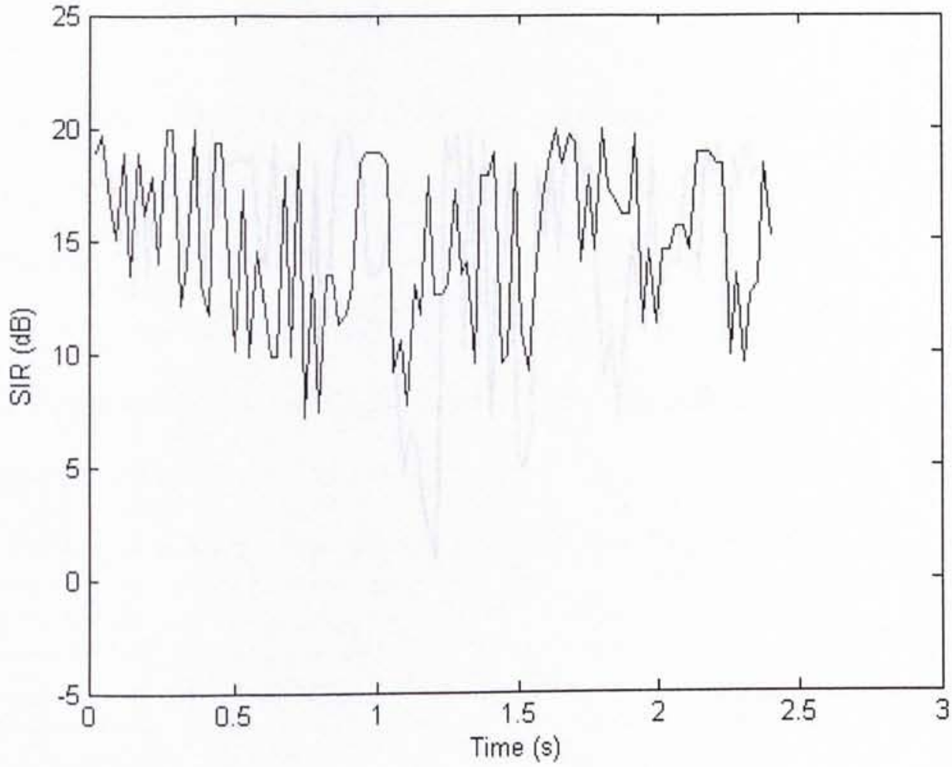

Figure 3.25: User 1's SIR variation (SMCT scheme)

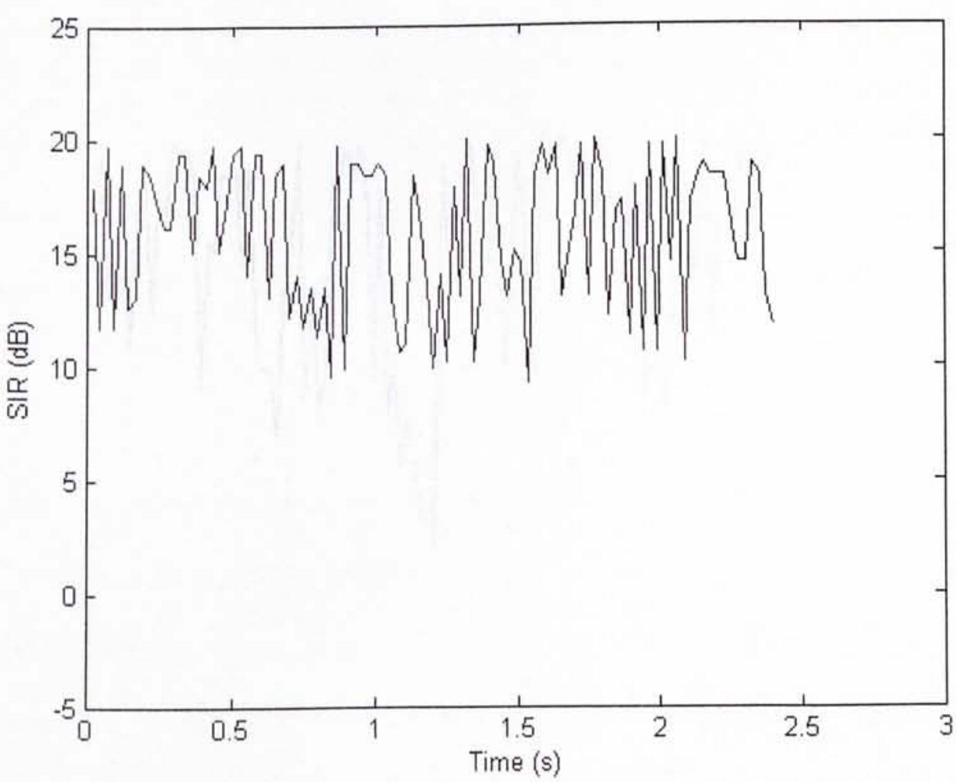

Figure 3.26: User 2's SIR variation (SMCT scheme)

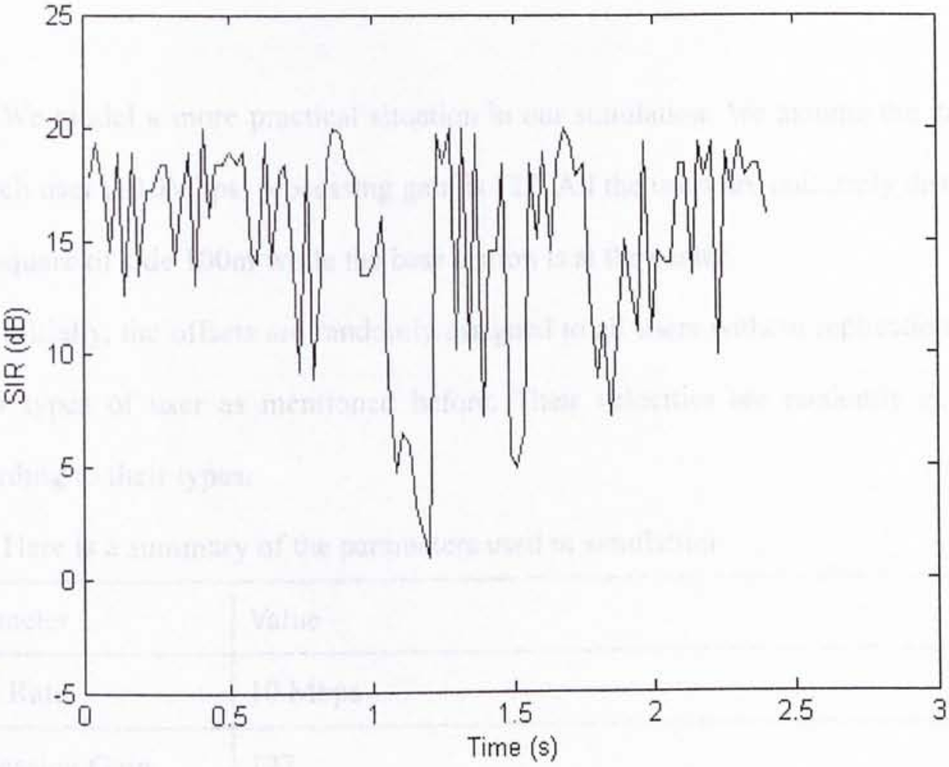

Figure 3.27: User 3's SIR variation (SMCT scheme)

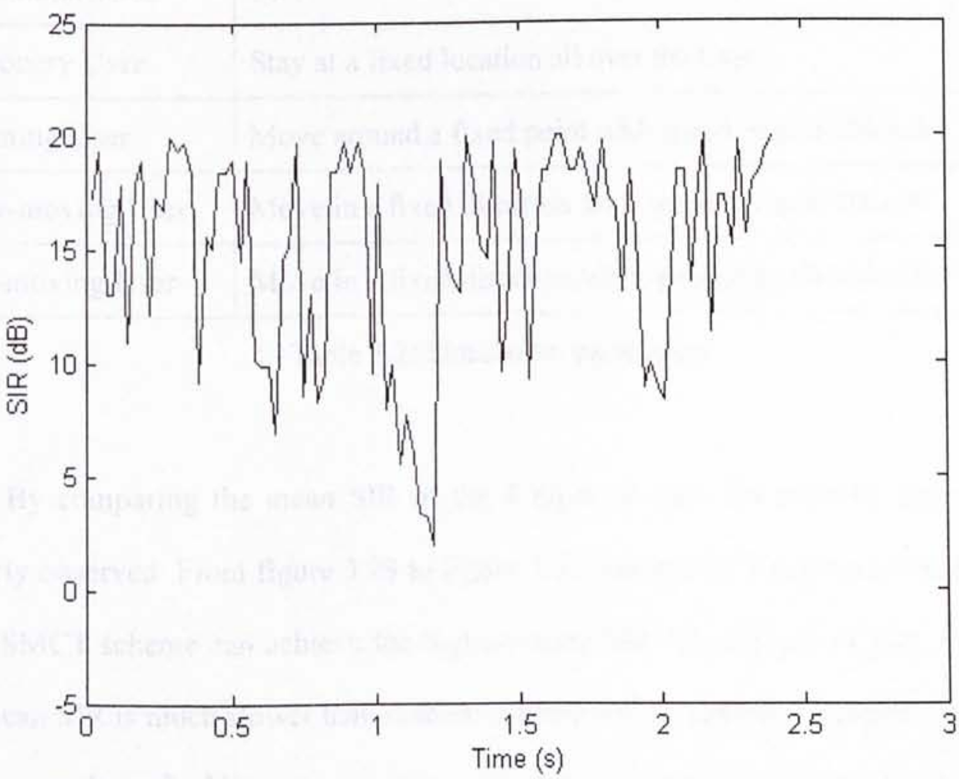

Figure 3.28: User 4's SIR variation (SMCT scheme)

3.2.3 Simulation Result

We model a more practical situation in our simulation. We assume the data rate of each user is 10Mbps, processing gain is 127. All the users are uniformly distributed in a square of side 100m while the base station is at the center.

Initially, the offsets are randomly assigned to all users without replication. There are 4 types of user as mentioned before. Their velocities are randomly generated according to their types.

Here is a summary of the parameters used in simulation:

Parameter	Value
Data Rate	10 Mbps
Processing Gain	127
Chip Rate	1270 Mcps
Simulation Time	56 s
Stationery User	Stay at a fixed location all over the time
Roaming User	Move around a fixed point with speed 3km/h-10km/h
Slow-moving User	Move in a fixed direction with speed 10km/h-30km/h
Fast-moving User	Move in a fixed direction with speed 20km/h-60km/h

Table 3.2: Simulation parameters

By comparing the mean SIR of the 4 types of user, the capacity gain can be clearly observed. From figure 3.29 to figure 3.32, among the 3 schemes, it is obvious that SMCT scheme can achieve the highest mean SIR for all types of user. The drop of mean SIR is much slower than random scheme and no scheme. Capacity gain refers to the number of additional user that the system can support in order to maintain a certain level of SIR. For example, for the stationery user, if the minimum acceptable mean SIR is 16 dB, the system employing no scheme can only support 13 users, however, the system employing SMCT (20Hz) scheme can support approximately 20 users.

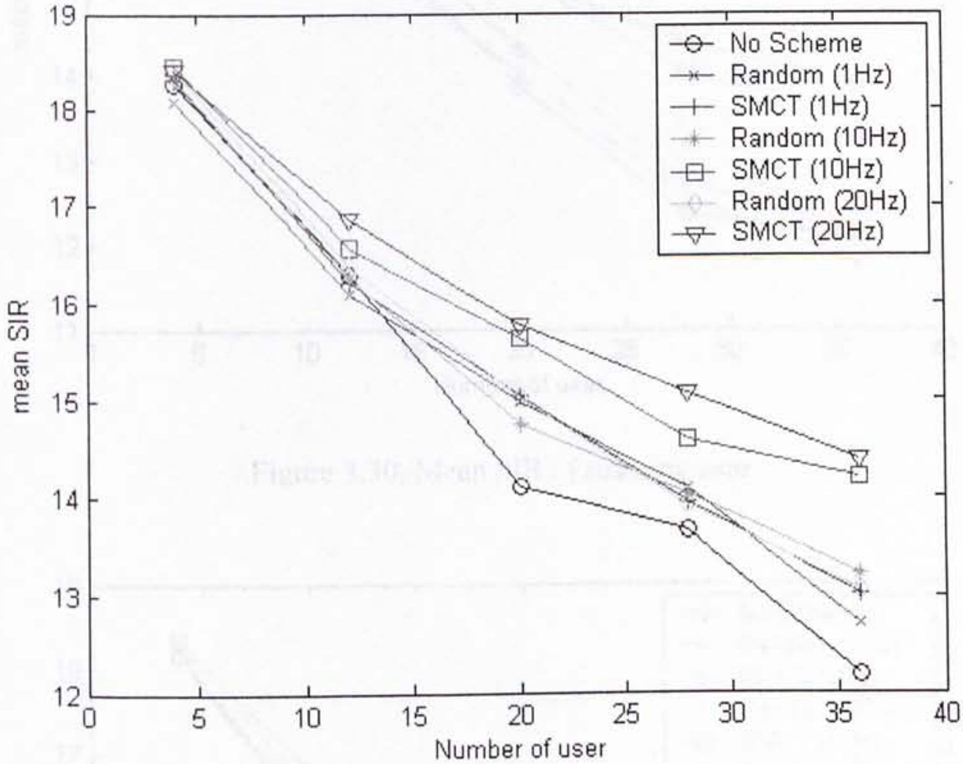

Figure 3.29: Mean SIR of stationery user

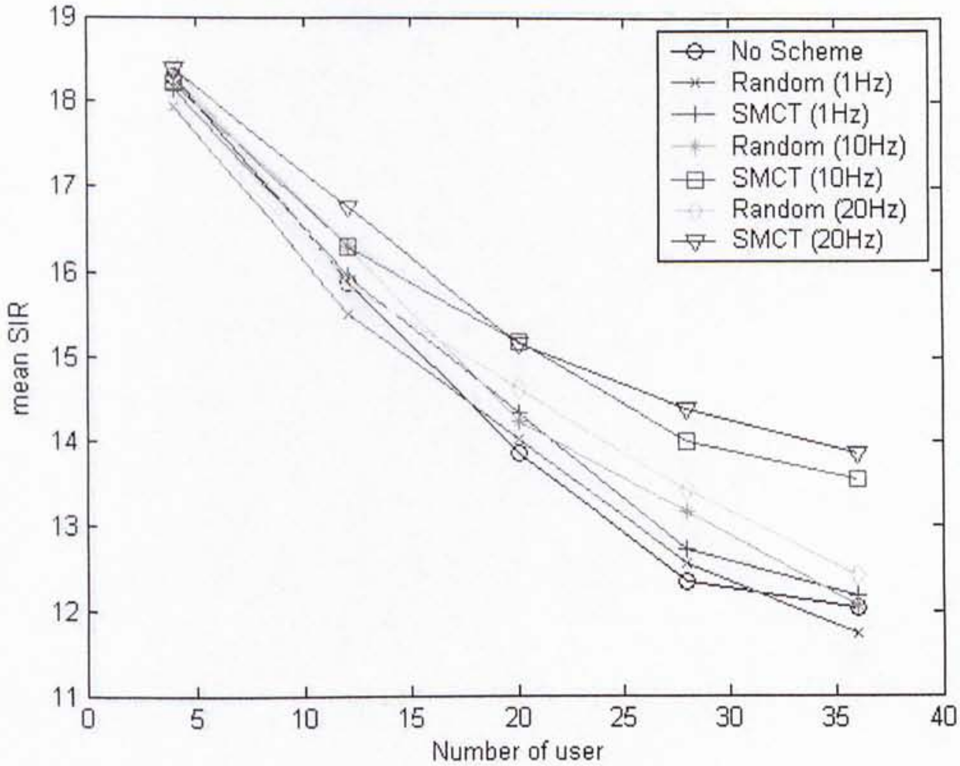

Figure 3.30: Mean SIR of roaming user

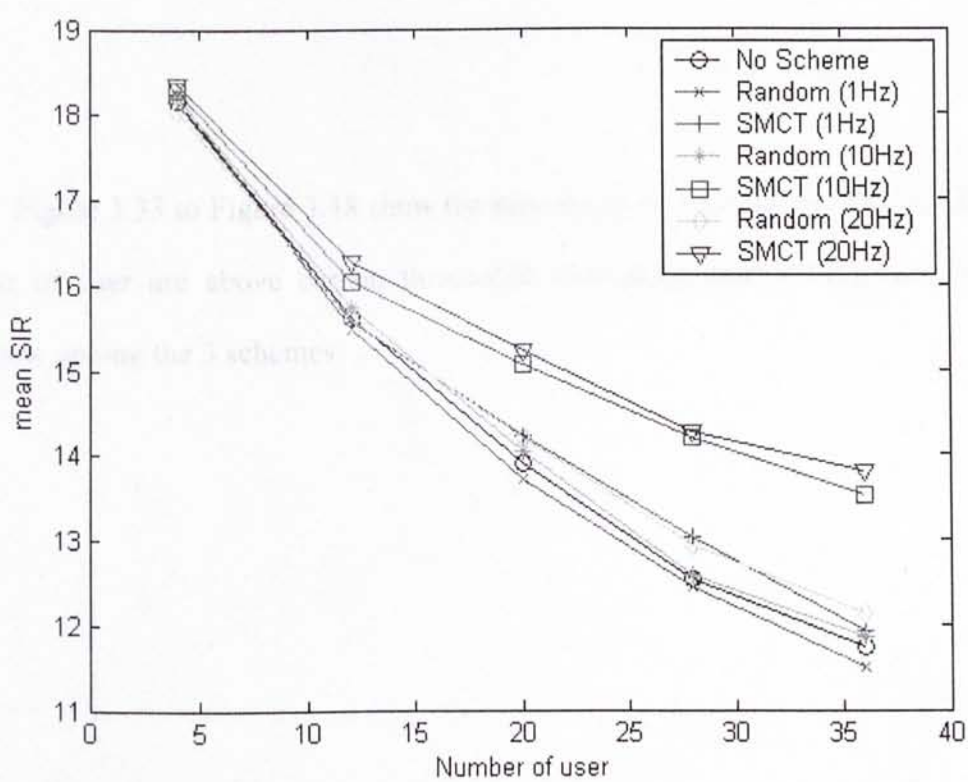

Figure 3.31: Mean SIR of slow-moving user

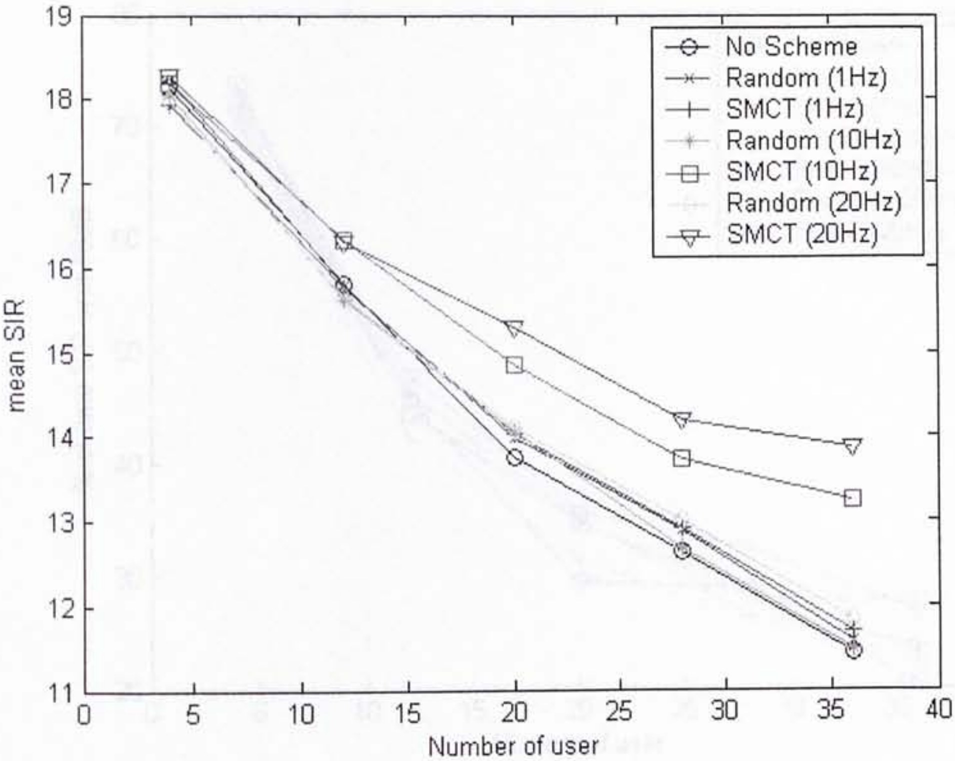

Figure 3.32: Mean SIR of fast-moving user

Figure 3.33 to Figure 3.48 show the percentage of time that the SIRs of different types of user are above certain thresholds. Obviously, SMCT is always the best scheme among the 3 schemes.

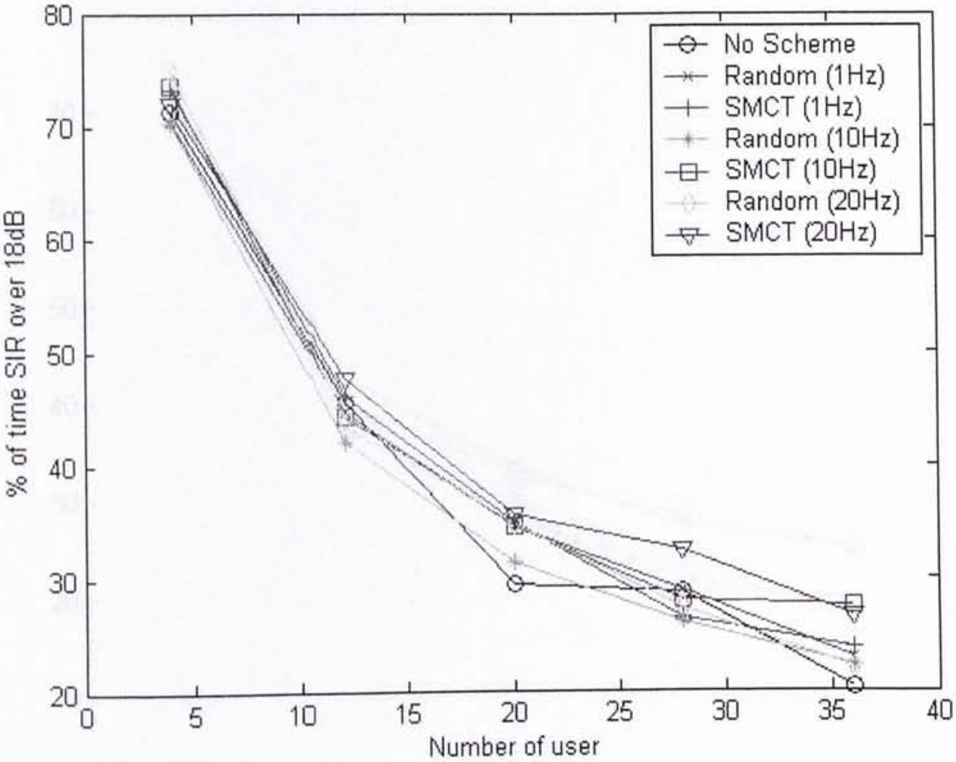

Figure 3.33: % of time that SIR over 18dB (stationery user)

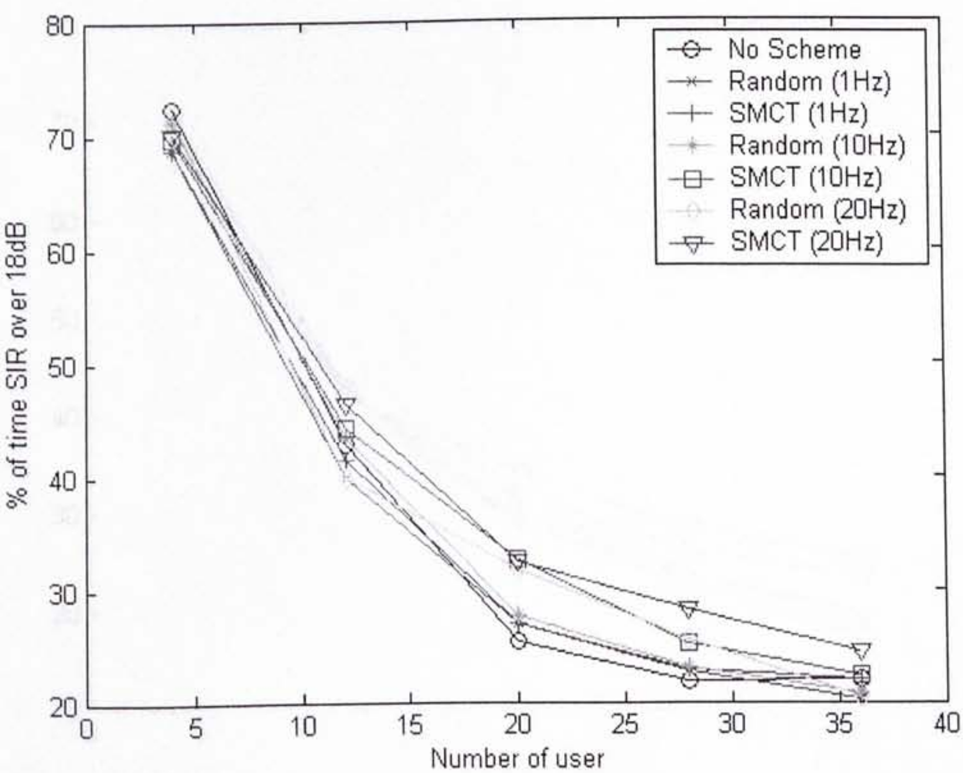

Figure 3.34: % of time that SIR over 18dB (roaming user)

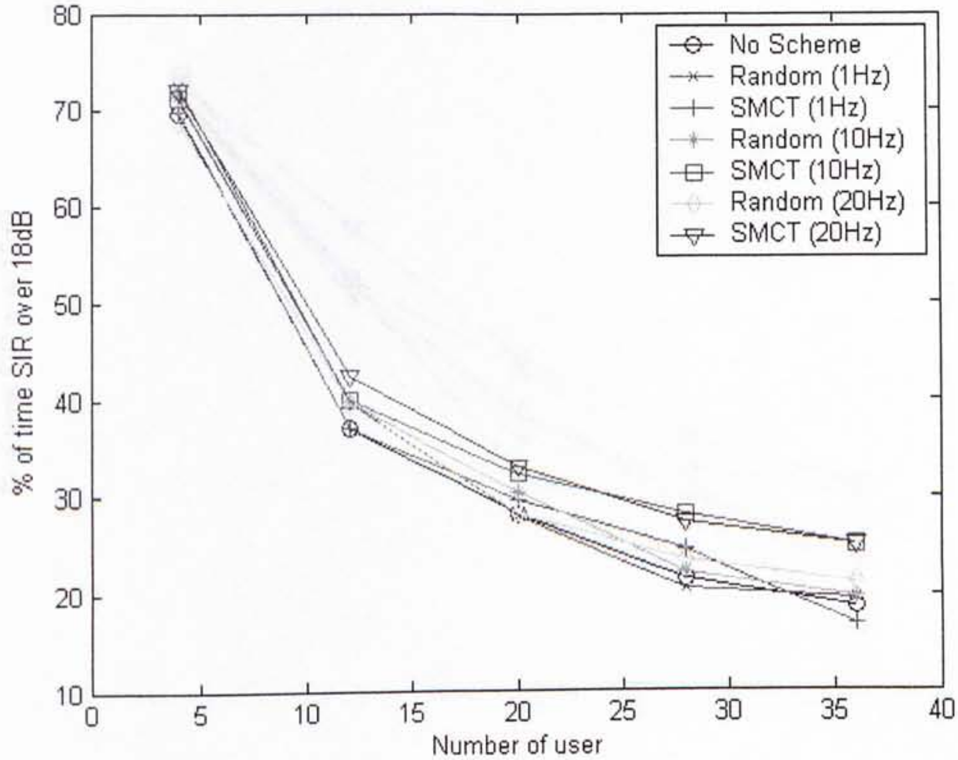

Figure 3.35: % of time that SIR over 18dB (slow-moving user)

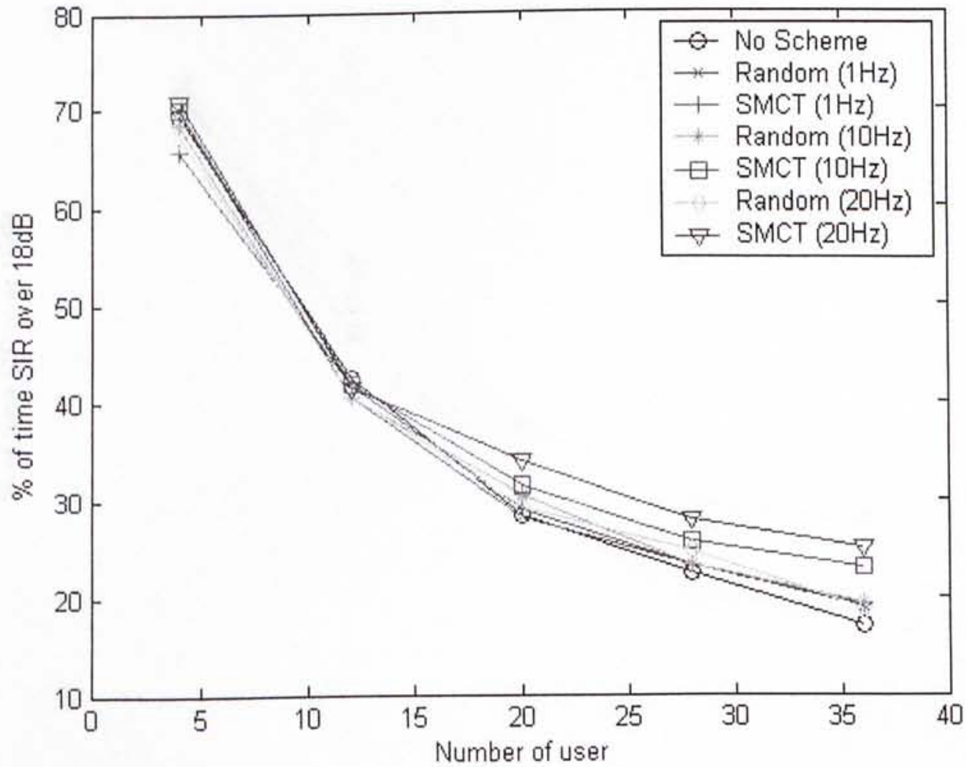

Figure 3.36: % of time that SIR over 18dB (fast-moving user)

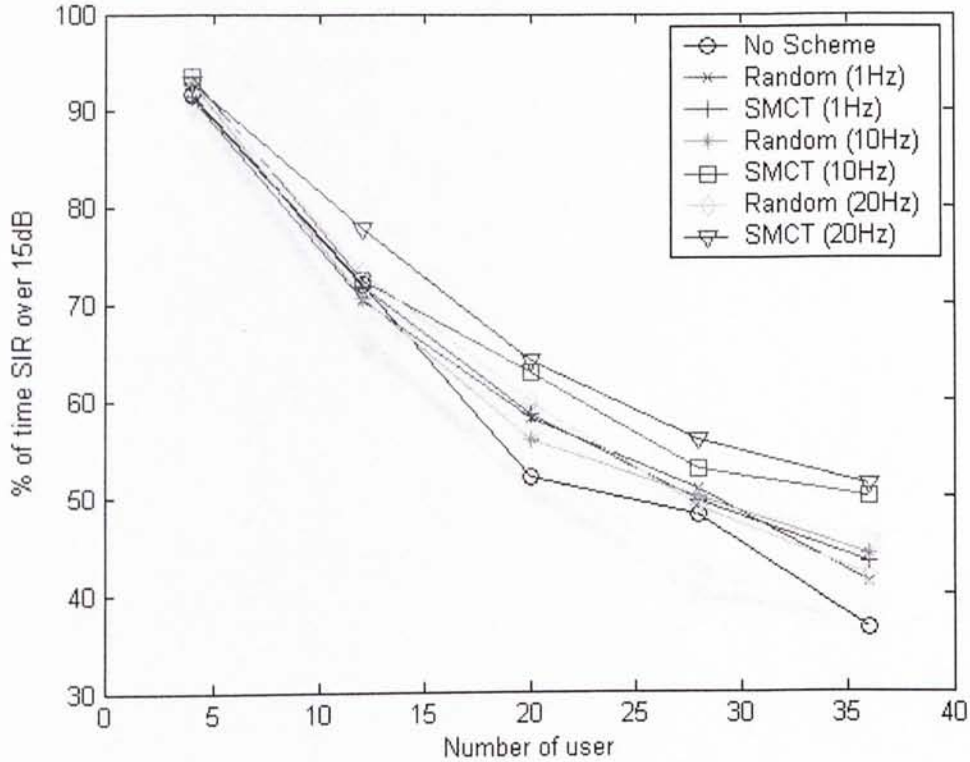

Figure 3.37: % of time that SIR over 15dB (stationery user)

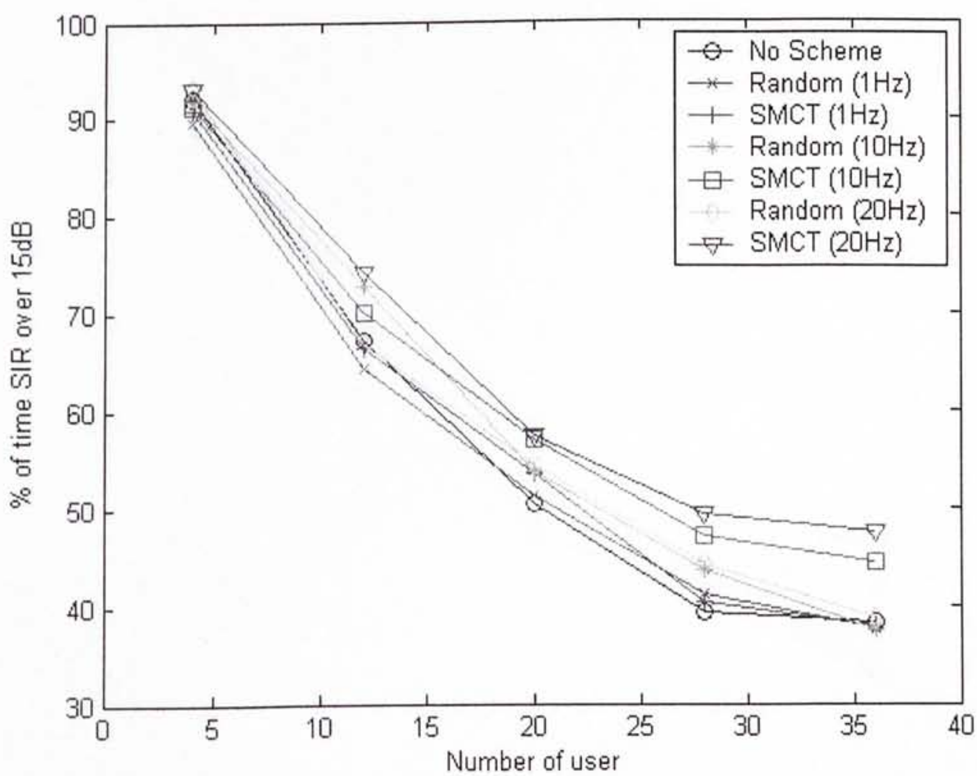

Figure 3.38: % of time that SIR over 15dB (roaming user)

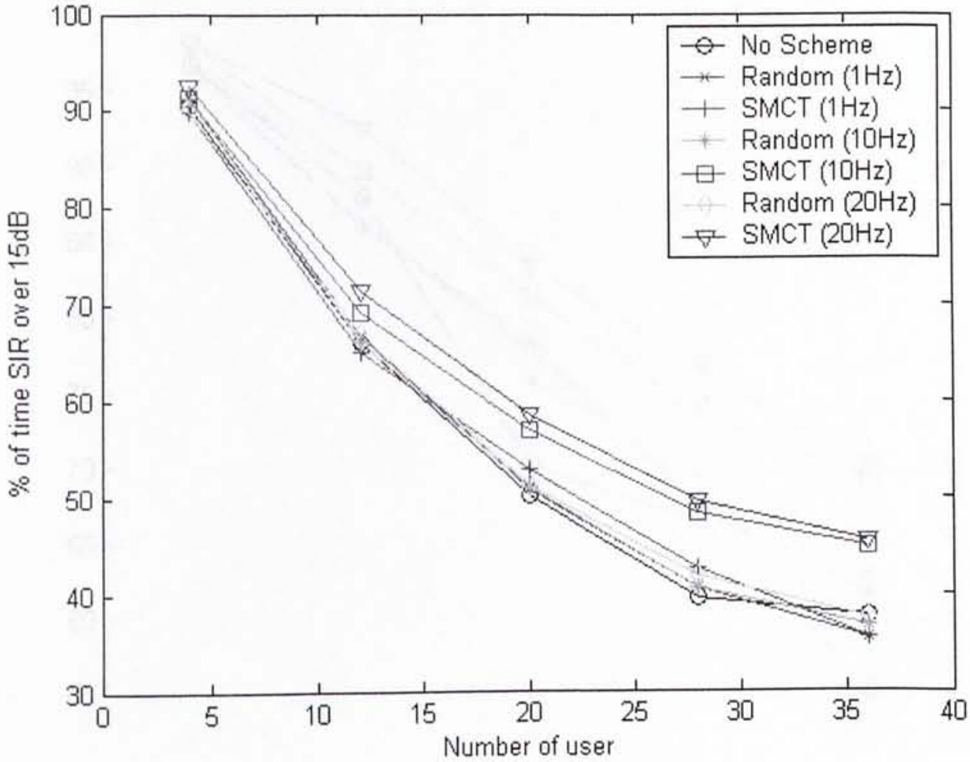

Figure 3.39: % of time that SIR over 15dB (slow-moving user)

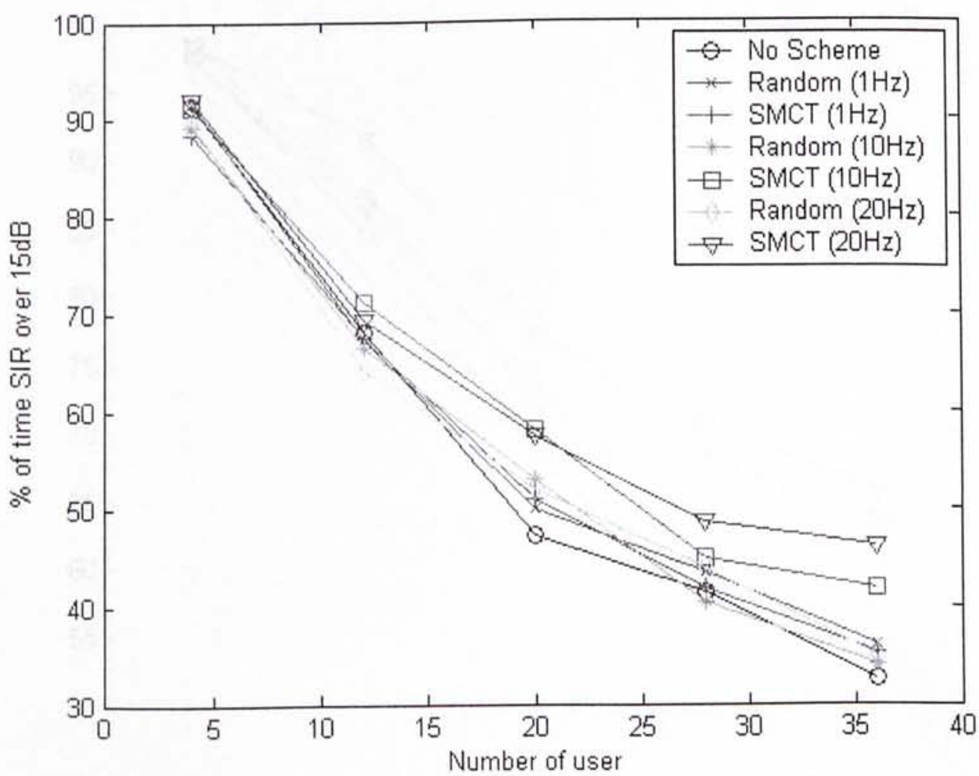

Figure 3.40: % of time that SIR over 15dB (fast-moving user)

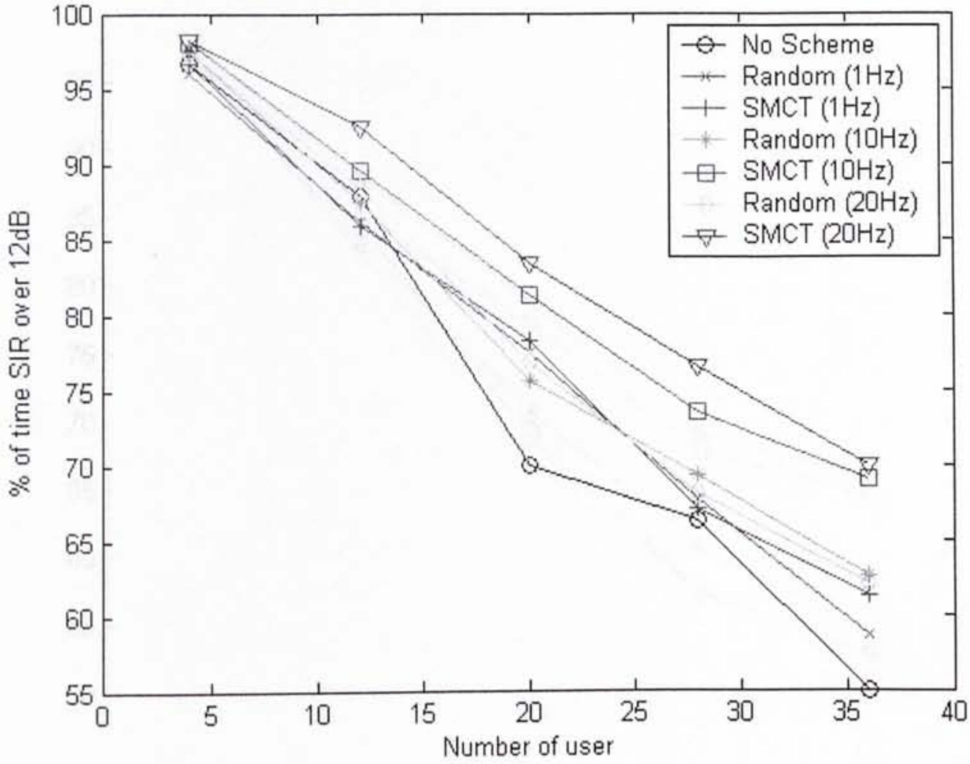

Figure 3.41: % of time that SIR over 12dB (stationery user)

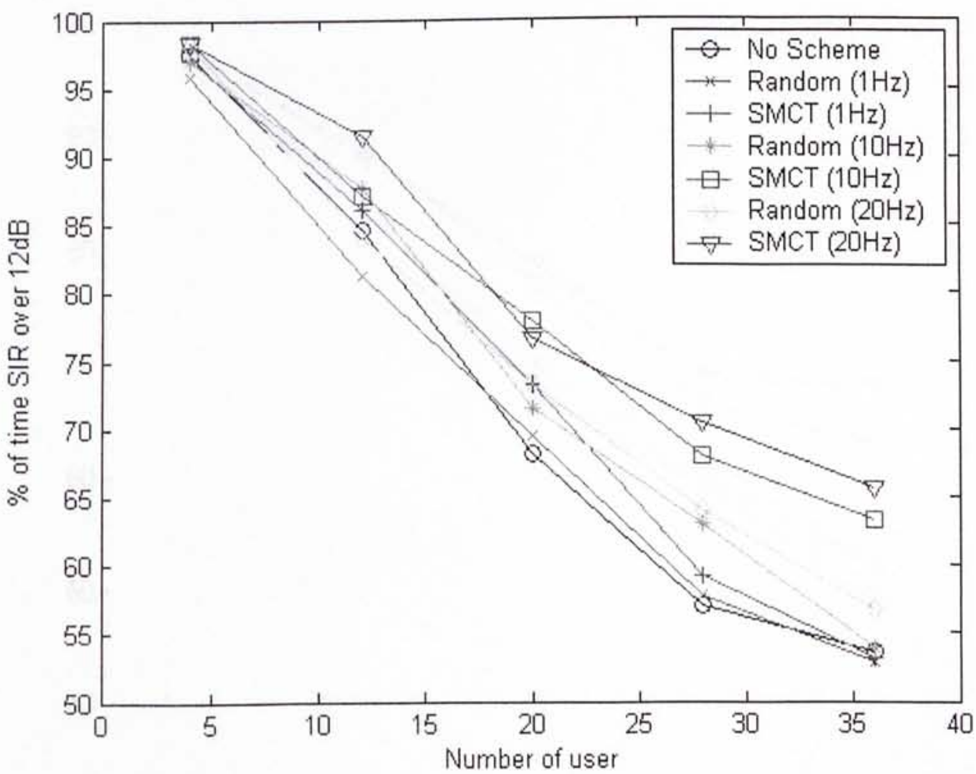

Figure 3.42: % of time that SIR over 12dB (roaming user)

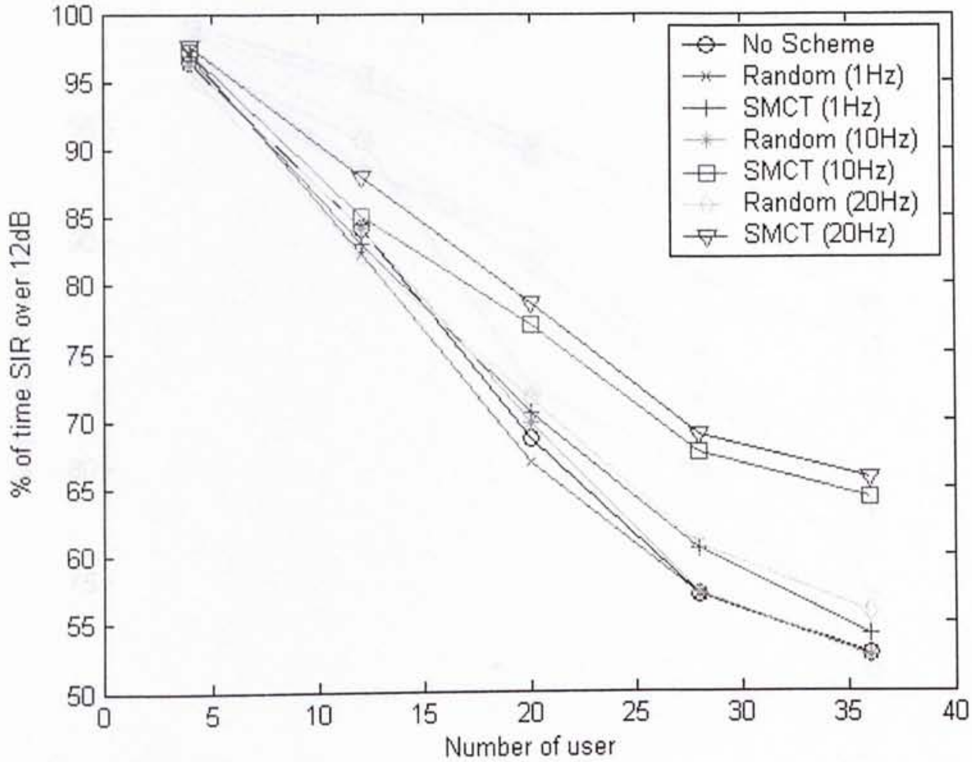

Figure 3.43: % of time that SIR over 12dB (slow-moving user)

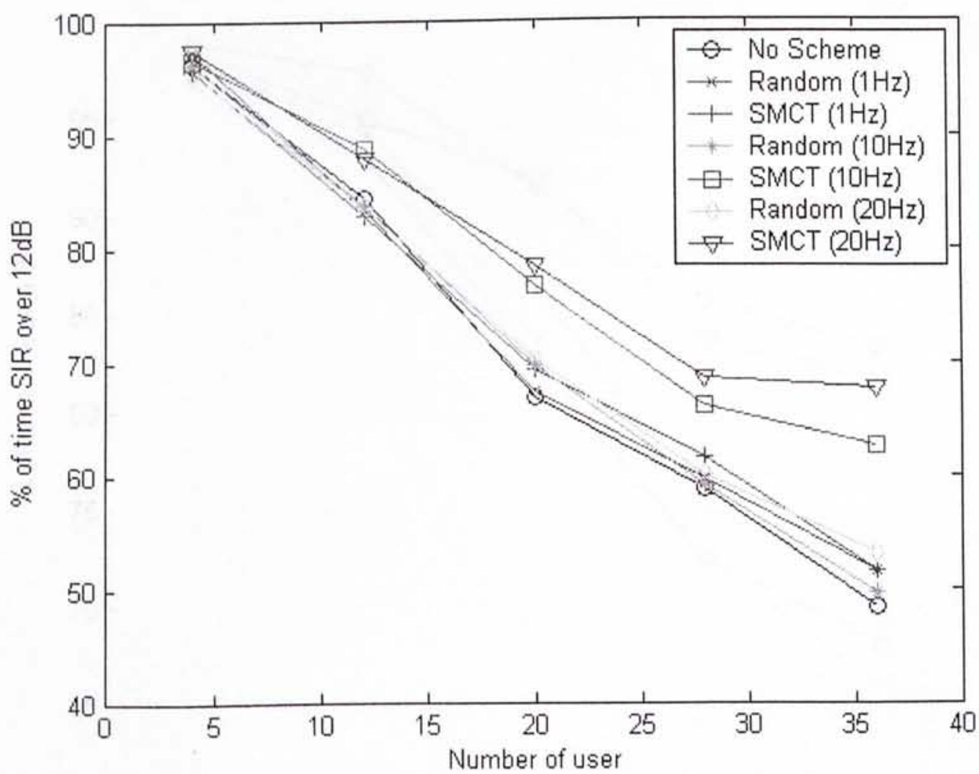

Figure 3.44: % of time that SIR over 12dB (fast-moving user)

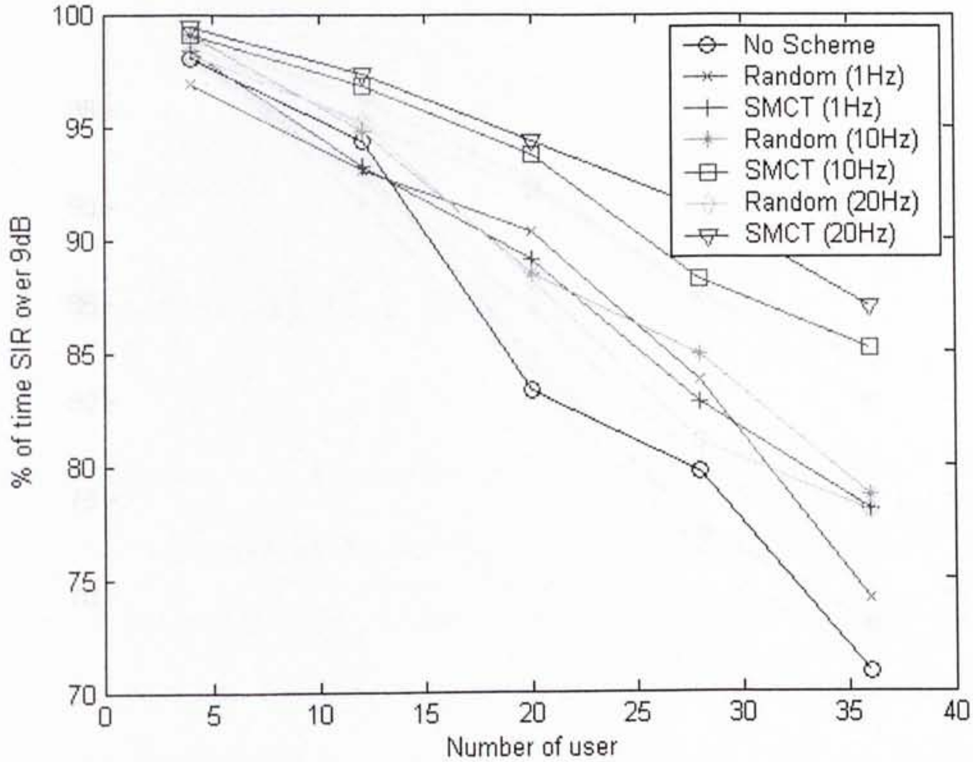

Figure 3.45: % of time that SIR over 9dB (stationery user)

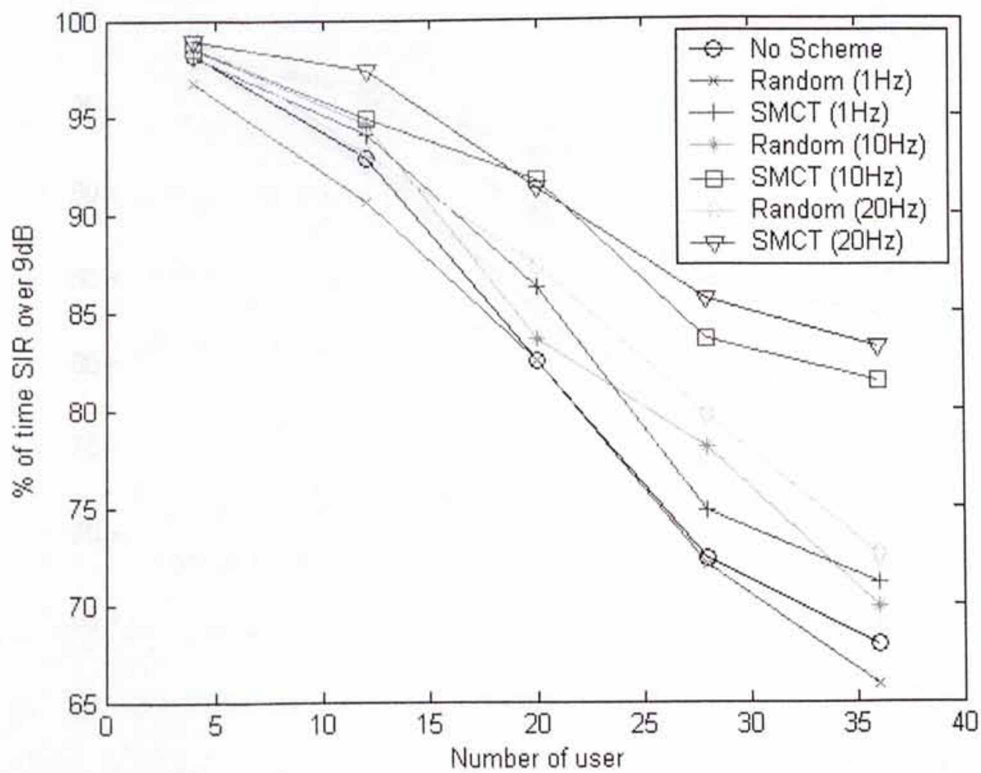

Figure 3.46: % of time that SIR over 9dB (roaming user)

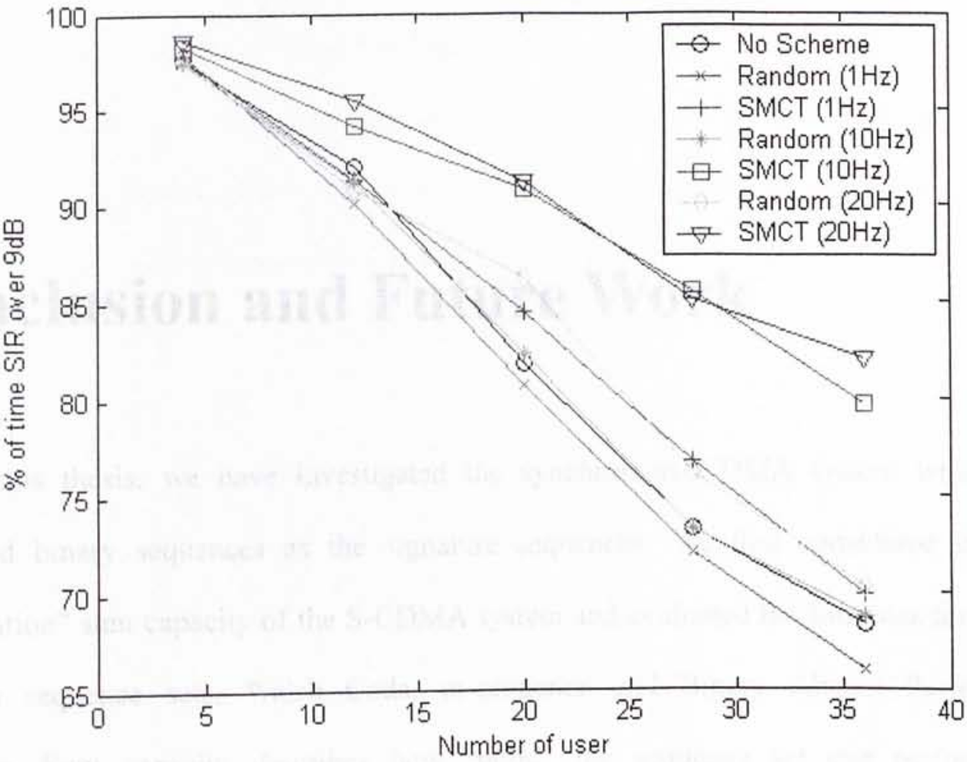

Figure 3.47: % of time that SIR over 9dB (slow-moving user)

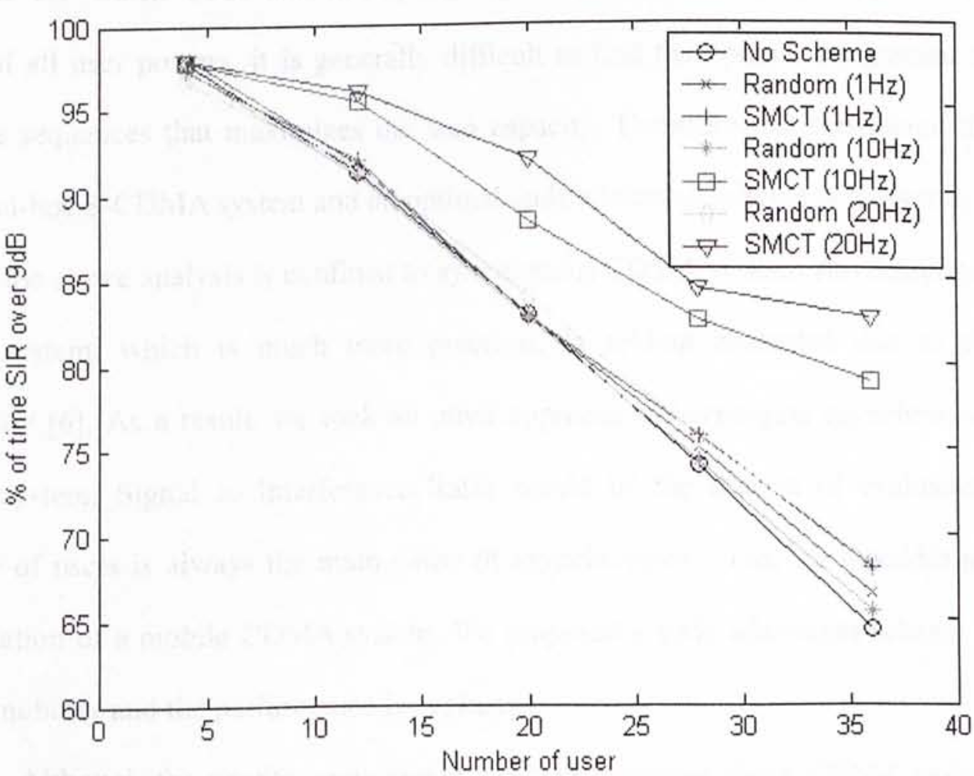

Figure 3.48: % of time that SIR over 9dB (fast-moving user)

Conclusion and Future Work

In this thesis, we have investigated the synchronous CDMA system which employed binary sequences as the signature sequences. We first considered the "information" sum capacity of the S-CDMA system and evaluated the formulas for 3 common sequence sets, Walsh Code, m-sequence and Binary Almost Perfect Sequence. Sum capacity describes how "good" the sequence set can perform ultimately or the "potential" of the sequence set. The asymptotic upper of the sum capacities for Walsh Code and m-sequence are also derived. However, given the profile of all user powers, it is generally difficult to find the optimal assignment of signature sequences that maximizes the sum capacity. Therefore, we considered the case of ad-hoc S-CDMA system and an optimal code allocation scheme is proposed.

All the above analysis is confined to synchronous CDMA system. Asynchronous CDMA system, which is much more practical, is seldom evaluated due to the complexity [6]. As a result, we took an other approach to investigate asynchronous CDMA system. Signal to Interference Ratio would be the criteria of evaluation. Mobility of users is always the main cause of asynchronism. Thus, we consider the SIR variation of a mobile CDMA system. We proposed a code adaptation scheme to combat mobility and the performance is evaluated.

Although the results complement existing literature about CDMA system which employs binary signature sequence, there are still some assumptions and restrictions preventing the results from applying to practical situation. The results about sum capacity are far from practical since synchronism between the mobile users is not always the case. Sum capacity formula for asynchronous CDMA system is always desirable. Besides, the binary sequence set that maximizes sum capacity of the S-CDMA system is also an interesting direction to explore.

Appendix

Lemma 1:

Let $\mathbf{D}$ be a $j \times j$ diagonal matrix, such that

$$\mathbf{D} = \begin{bmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & a_j \end{bmatrix}$$

Then

$$\det\left(\mathbf{D} + \mathbf{A}_j(r)\right) = \det\begin{bmatrix} r+a_1 & r & \dots & r \\ r & r+a_2 & & r \\ \vdots & & \ddots & \vdots \\ r & r & \dots & r+a_j \end{bmatrix} = r\prod_{i=1}^{j} a_i\left[\frac{1}{r} + \sum_{i=1}^{j}\frac{1}{a_i}\right]$$

Proof of Lemma 1:

$$\begin{aligned}
\det\left(\mathbf{D} + \mathbf{A}_j(r)\right) &= \det\left[\mathbf{D}\left(\mathbf{I} + \mathbf{D}^{-1}\mathbf{A}_j(r)\right)\right] \\
&= \det\mathbf{D}\det\left(\mathbf{I} + \mathbf{D}^{-1}\mathbf{A}_j(r)\right) \\
&= \det\mathbf{D}\det\left(\mathbf{I} + \mathbf{D}^{-1}\underline{\mathbf{c}}_j(r)\cdot\underline{\mathbf{c}}_j(1)^T\right) \\
&= \det\mathbf{D}\left[1 + \underline{\mathbf{c}}_j(1)^T\,\mathbf{D}^{-1}\underline{\mathbf{c}}_j(r)\right] \\
&= \prod_{i=1}^{j} a_i\left[1 + \sum_{i=1}^{j}\frac{r}{a_i}\right] \\
&= r\prod_{i=1}^{j} a_i\left[\frac{1}{r} + \sum_{i=1}^{j}\frac{1}{a_i}\right]
\end{aligned}$$

Lemma 2:

Let $\mathbf{H}$ be a $n \times n$ square matrix, define the operation $SUM(\mathbf{H})$ be the sum of all n^2 elements. Let $\mathbf{D}$ be a $j \times j$ diagonal matrix, such that

$$\mathbf{D} = \begin{bmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & a_j \end{bmatrix}$$

Then

$$SUM\left[\left(\mathbf{D} + \mathbf{A}_j(r)\right)^{-1}\right] = SUM\left(\begin{bmatrix} r + a_1 & r & \dots & r \\ r & r + a_2 & & r \\ \vdots & & \ddots & \vdots \\ r & r & \dots & r + a_j \end{bmatrix}^{-1}\right) = \frac{\sum_{i=1}^{j} \frac{1}{a_i}}{1 + r\sum_{i=1}^{j} \frac{1}{a_i}}$$

Proof of Lemma 2:

Let $\tilde{\mathbf{D}} = \mathbf{D} + \mathbf{A}_j(r) = \begin{bmatrix} r + a_1 & r & \dots & r \\ r & r + a_2 & & r \\ \vdots & & \ddots & \vdots \\ r & r & \dots & r + a_j \end{bmatrix}$

By using Gauss-Jordon Elimination to find the inverse:

$\langle \tilde{\mathbf{D}} \mid \mathbf{I} \rangle$

$$\left\langle \begin{array}{cccc|cccc} r + a_1 & r & \cdots & r & 1 & 0 & \cdots & 0 \\ r & r + a_2 & & r & 0 & 1 & & 0 \\ \vdots & & \ddots & \vdots & \vdots & & \ddots & \vdots \\ r & r & \cdots & r + a_j & 0 & 0 & \cdots & 1 \end{array} \right\rangle$$

Replace $Row(k)$ by $Row(k) - Row(k+1)$ for $k = 1, 2, \cdots, j-1$.

We have,

$$
\left(\begin{array}{cccccccc|cccccccc}
a_1 & -a_2 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & a_2 & -a_3 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & -1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & a_3 & -a_4 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 & -1 & 0 & \cdots & 0 & 0 \\
\vdots & & & \ddots & \ddots & & & \vdots & \vdots & & & & \ddots & \ddots & & \vdots \\
\vdots & & & \ddots & \ddots & & & \vdots & \vdots & & & & \ddots & \ddots & & \vdots \\
0 & 0 & 0 & \cdots & 0 & a_{j-2} & -a_{j-1} & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & a_{j-1} & -a_j & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -1 \\
r & r & r & r & r & \cdots & r & r+a_j & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1
\end{array}\right)
$$

Then eliminate the first element in the last row by the first row, second element by the second row and so on so forth,

$$
\left(\begin{array}{cccccccc|cccccccc}
a_1 & -a_2 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & a_2 & -a_3 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & -1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & a_3 & -a_4 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 & -1 & 0 & \cdots & 0 & 0 \\
\vdots & & & \ddots & \ddots & & & \vdots & \vdots & & & \ddots & \ddots & & & \vdots \\
\vdots & & & \ddots & \ddots & & & \vdots & \vdots & & & & \ddots & \ddots & & \vdots \\
0 & 0 & 0 & \cdots & 0 & a_{j-2} & -a_{j-1} & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & a_{j-1} & -a_j & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & a_j + r a_j \sum_{i=1}^{j} \frac{1}{a_i} & -\frac{r}{a_1} & -\frac{r}{a_2} & -\frac{r}{a_3} & -\frac{r}{a_4} & -\frac{r}{a_5} & \cdots & -\frac{r}{a_{j-1}} & 1+r\sum_{i=1}^{j-1}\frac{1}{a_i}
\end{array}\right)
$$

Let $\beta = a_j + r a_j \sum_{i=1}^{j} \frac{1}{a_i}$,

Divide $Row(k)$ by a_k for $k = 1, 2, \cdots, j-1$ and divide $Row(j)$ by β

$$
\left(\begin{array}{cccccccc|cccccccc}
1 & -\frac{a_2}{a_1} & 0 & 0 & 0 & \cdots & 0 & 0 & \frac{1}{a_1} & -\frac{1}{a_1} & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 & -\frac{a_3}{a_2} & 0 & 0 & \cdots & 0 & 0 & 0 & \frac{1}{a_2} & -\frac{1}{a_2} & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & -\frac{a_4}{a_3} & 0 & \cdots & 0 & 0 & 0 & 0 & \frac{1}{a_3} & -\frac{1}{a_3} & 0 & \cdots & 0 & 0 \\
\vdots & & & \ddots & \ddots & & & \vdots & \vdots & & & \ddots & \ddots & & & \vdots \\
\vdots & & & \ddots & \ddots & & & \vdots & \vdots & & & & \ddots & \ddots & & \vdots \\
0 & 0 & 0 & \cdots & 0 & 1 & -\frac{a_{j-1}}{a_{j-2}} & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{a_{j-2}} & -\frac{1}{a_{j-2}} & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 1 & -\frac{a_j}{a_{j-1}} & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{a_{j-1}} & -\frac{1}{a_{j-1}} \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -\frac{r}{\beta a_1} & -\frac{r}{\beta a_2} & -\frac{r}{\beta a_3} & -\frac{r}{\beta a_4} & -\frac{r}{\beta a_5} & \cdots & -\frac{r}{\beta a_{j-1}} & \frac{1}{\beta}+r\sum_{i=1}^{j-1}\frac{1}{\beta a_i}
\end{array}\right)
$$

Let it be $\langle \mathbf{M} \mid \mathbf{N} \rangle$

The remaining steps are replacing $Row(k)$ by $Row(k)+\frac{a_{k+1}}{a_k}Row(k+1)$ for $k=1,2,\cdots,j-1$ in order to convert the matrix $\mathbf{M}$ into $\mathbf{I}$ and the matrix $\mathbf{N}$ into $\tilde{\mathbf{D}}^{-1}$.

However, what we concern is the sum of elements of the inverse.

Define $RS(\mathbf{C},i)$ be the sum of j elements of i-th row of matrix $\mathbf{C}$. We have

$$RS(\mathbf{N},k)=\frac{1}{a_k}-\frac{1}{a_k}=0 \text{ for } k=1,2,\cdots,j-1$$

$$\begin{aligned} RS(\mathbf{N},j) &= -\frac{r}{\beta a_1}-\frac{r}{\beta a_2}-\frac{r}{\beta a_3}-\cdots-\frac{r}{\beta a_{j-1}}+\frac{1}{\beta}+r\sum_{i=1}^{j-1}\frac{1}{\beta a_i} \\ &= -\sum_{i=1}^{j-1}\frac{1}{\beta a_i}+\frac{1}{\beta}+r\sum_{i=1}^{j-1}\frac{1}{\beta a_i} \\ &= \frac{1}{\beta} \end{aligned}$$

And $SUM\left(\tilde{\mathbf{D}}^{-1}\right)$ can be expressed as:

$$SUM\left(\tilde{\mathbf{D}}^{-1}\right)=\sum_{t=1}^{j}RS\left(\tilde{\mathbf{D}}^{-1},t\right).$$

The last operation, replacing $Row(k)$ by $Row(k)+\frac{a_{k+1}}{a_k}Row(k+1)$ for $k=1,2,\cdots,j-1$, has the effect on Row Sum as following:

$$RS\left(\tilde{\mathbf{D}}^{-1},k\right)=RS(\mathbf{N},k)+\frac{a_{k+1}}{a_k}RS\left(\tilde{\mathbf{D}}^{-1},k+1\right) \text{ for } k=1,2,\cdots,j-1$$

$$RS\left(\tilde{\mathbf{D}}^{-1},j\right)=RS(\mathbf{N},j)$$

We have:

$$RS\left(\tilde{\mathbf{D}}^{-1},k\right)=\frac{a_{k+1}}{a_k}RS\left(\tilde{\mathbf{D}}^{-1},k+1\right) \text{ for } k=1,2,\cdots,j-1$$

$$RS\left(\tilde{\mathbf{D}}^{-1},j\right)=\frac{1}{\beta}$$

As a result, we can find $RS\left(\tilde{\mathbf{D}}^{-1},k\right)$ for all $k=1,2,\cdots,j$

$$RS\left(\tilde{\mathbf{D}}^{-1},k\right)=\frac{1}{\beta}\frac{a_j}{a_k} \text{ for } k=1,2,\ldots,j$$

Therefore,

$$SUM(\tilde{\mathbf{D}}^{-1})=\sum_{i=1}^{j}RS\left(\tilde{\mathbf{D}}^{-1},i\right)=\frac{a_j}{\beta}\sum_{i=1}^{j}\frac{1}{a_i}$$

Put $\beta=a_j+ra_j\sum_{i=1}^{j}\frac{1}{a_i}$

$$SUM(D^{-1})=\frac{a_j\sum_{i=1}^{j}\frac{1}{a_i}}{a_j+ra_j\sum_{i=1}^{j}\frac{1}{a_i}}$$

$$=\frac{\sum_{i=1}^{j}\frac{1}{a_i}}{1+r\sum_{i=1}^{j}\frac{1}{a_i}}$$

Proposition 4:

$$\begin{aligned} &\text{maximize} && \frac{1}{L}\log\left\{\prod_{j=1}^{L}\left[1+\frac{LP_{(j)}}{N_o}\right]\right\} \\ &\text{subject to} && P_{(1)}+P_{(2)}+\cdots+P_{(L)}=P_{tot} \\ &&& P_{(i)}\geq 0,\ P_{(i)}\in\Re \qquad \forall i=1,2,\cdots,L \end{aligned}$$

The solution of above optimization problem is $P_{(j)}=\dfrac{P_{tot}}{L}\quad \forall j=1,2,\cdots,L$ and the maximum value is $\log\left(1+\dfrac{P_{tot}}{N_o}\right)$.

Proof of proposition 4:

For convenience, we change the variable from $P_{(j)}$ to x_j, P_{tot} to X_T, so the optimization problem becomes:

$$\begin{aligned} &\text{maximize} && \frac{1}{L}\log\left\{\prod_{j=1}^{L}\left[1+\frac{Lx_j}{N_o}\right]\right\} \\ &\text{subject to} && x_1+x_2+\cdots+x_L=X_T \\ &&& x_i\geq 0 \qquad \forall i=1,2,\cdots,L \end{aligned}$$

which is equivalent to

$$\begin{aligned} &\text{minimize} && -\frac{1}{L}\log\left\{\prod_{j=1}^{L}\left[1+\frac{Lx_j}{N_o}\right]\right\} \\ &\text{subject to} && x_1+x_2+\cdots+x_L=X_T \\ &&& x_i\geq 0 \qquad \forall i=1,2,\cdots,L \end{aligned}$$

By using the Method of Lagrangian Multiplier,

Let $f(\underline{\mathbf{x}})=-\dfrac{1}{L}\log\left\{\prod_{j=1}^{L}\left[1+\dfrac{Lx_j}{N_o}\right]\right\} \qquad \underline{\mathbf{x}}=[x_1,x_2,\cdots,x_L]^T$

$$\nabla f(\underline{\mathbf{x}}) = \begin{bmatrix} \dfrac{\partial f(\underline{\mathbf{x}})}{\partial x_1} \\ \dfrac{\partial f(\underline{\mathbf{x}})}{\partial x_2} \\ \vdots \\ \dfrac{\partial f(\underline{\mathbf{x}})}{\partial x_L} \end{bmatrix} = \begin{bmatrix} -\dfrac{1}{N_o + Lx_1} \\ -\dfrac{1}{N_o + Lx_2} \\ \vdots \\ -\dfrac{1}{N_o + Lx_L} \end{bmatrix} \quad \text{and} \quad \nabla h(\underline{\mathbf{x}}) = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

Let $\underline{\mathbf{x}}^*$ be the local minimum of f.

By the first order necessary condition:

$$\nabla f(\underline{\mathbf{x}}^*) + \lambda^* \nabla h(\underline{\mathbf{x}}^*) = 0$$

$$\Rightarrow \begin{cases} -\dfrac{1}{N_o + Lx_1^*} + \lambda^* = 0 \\ -\dfrac{1}{N_o + Lx_2^*} + \lambda^* = 0 \\ \qquad \vdots \\ -\dfrac{1}{N_o + Lx_L^*} + \lambda^* = 0 \end{cases}$$

$$\Rightarrow x_i^* = \frac{1}{L}\left(\frac{1}{\lambda^*} - N_o\right) \qquad \forall i = 1, 2, \cdots, L$$

$$\Rightarrow x_1^* = x_2^* = \cdots = x_L^* = \frac{X_T}{L}$$

The Hessian of the Lagrangian is:

$$\nabla_{xx}^2 L(\underline{\mathbf{x}}^*, \lambda^*) = \nabla^2 f(\underline{\mathbf{x}}^*) + \lambda^* \nabla^2 h(\underline{\mathbf{x}}^*)$$

Since

$$\frac{\partial^2 f(\underline{\mathbf{x}}^*)}{\partial x_i \partial x_k} = 0 \qquad \forall i \neq k$$

$$\frac{\partial^2 f(\underline{\mathbf{x}}^*)}{\partial x_i^2} = \frac{L}{\left(N_o + LX_T\right)^2} \qquad \forall i = 1, 2, \cdots, L$$

We have

$$\nabla^2 f(\underline{\mathbf{x}}^*) = diag\left(\frac{\partial^2 f(\underline{\mathbf{x}}^*)}{\partial x_1^{\ 2}}, \frac{\partial^2 f(\underline{\mathbf{x}}^*)}{\partial x_2^{\ 2}}, \cdots, \frac{\partial^2 f(\underline{\mathbf{x}}^*)}{\partial x_L^{\ 2}}\right)$$

$$= diag\left(\frac{L}{(N_o + LX_T)^2}, \frac{L}{(N_o + LX_T)^2}, \cdots, \frac{L}{(N_o + LX_T)^2}\right)$$

$$= \frac{L}{(N_o + LX_T)^2}\mathbf{I_L}$$

Thus,

$$\nabla_{xx}^2 L(\underline{\mathbf{x}}^*, \lambda^*) = \frac{L}{(N_o + LX_T)^2}\mathbf{I_L} \qquad \text{since } \nabla^2 h(\underline{\mathbf{x}}^*) = \mathbf{0}$$

To test whether the point $\underline{\mathbf{x}}^* = \left(\frac{X_T}{L}, \frac{X_T}{L}, \cdots, \frac{X_T}{L}\right)$ is a strict local minimum.

Consider

$$\forall \underline{\mathbf{y}} \in V = \left\{\underline{\mathbf{y}} \mid \nabla h(\underline{\mathbf{x}})^T \underline{\mathbf{y}} = 0\right\} = \left\{\underline{\mathbf{y}} \mid y_1 + y_2 + \cdots + y_L = 0\right\} \text{ with } \underline{\mathbf{y}} \neq 0.$$

$$\underline{\mathbf{y}}^T \nabla_{xx}^2 L(\underline{\mathbf{x}}^*, \lambda^*)\underline{\mathbf{y}} = \underline{\mathbf{y}}^T\left(\frac{L}{(N_o + LX_T)^2}\mathbf{I_L}\right)\underline{\mathbf{y}}$$

$$= \frac{L}{(N_o + LX_T)^2}\underline{\mathbf{y}}^T \cdot \underline{\mathbf{y}}$$

$$= \frac{L}{(N_o + LX_T)^2}\left(y_1^{\ 2} + y_2^{\ 2} + \cdots + y_L^{\ 2}\right)$$

$$> 0$$

Therefore, the point $\underline{\mathbf{x}}^* = \left(\frac{X_T}{L}, \frac{X_T}{L}, \cdots, \frac{X_T}{L}\right)$ is a strict local minimum.

By changing the variables back from x_j to $P_{(j)}$, X_T to P_{tot}, and substitute the optimal solution to the sum capacity formula, the result follows.

Proposition 5:

$$\begin{aligned}
&\text{maximize} && \frac{1}{L}\log\left\{\prod_{j=1}^{L}\left[1+\frac{(L+1)P_{(j)}}{N_o}\right]\cdot\left(1-\sum_{i=1}^{L}\frac{P_{(i)}}{N_o+(L+1)P_{(i)}}\right)\right\} \\
&\text{subject to} && P_{(1)}+P_{(2)}+\cdots+P_{(L)}=P_{tot} \\
& && P_{(i)}\geq 0,\ P_{(i)}\in\Re \qquad \forall i=1,2,\cdots,L
\end{aligned}$$

The solution of above optimization problem is $P_{(j)}=\dfrac{P_{tot}}{L}$ $\forall j=1,2,\cdots,L$ and the maximum value is $\log\left(1+\dfrac{(L+1)P_{tot}}{LN_o}\right)+\dfrac{1}{L}\log\left[\dfrac{LN_o+P_{tot}}{LN_o+(L+1)P_{tot}}\right]$.

Proof of proposition 5:

For convenience, we change the variable from $P_{(j)}$ to x_j, P_{tot} to X_T, so the optimization problem becomes:

$$\begin{aligned}
&\text{maximize} && \frac{1}{L}\log\left\{\prod_{j=1}^{L}\left[1+\frac{(L+1)x_j}{N_o}\right]\cdot\left(1-\sum_{i=1}^{L}\frac{x_i}{N_o+(L+1)x_i}\right)\right\} \\
&\text{subject to} && x_1+x_2+\cdots+x_L=X_T \\
& && x_i\geq 0 \qquad \forall i=1,2,\cdots,L
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&\text{minimize} && -\frac{1}{L}\log\left\{\prod_{j=1}^{L}\left[1+\frac{(L+1)x_j}{N_o}\right]\cdot\left(1-\sum_{i=1}^{L}\frac{x_i}{N_o+(L+1)x_i}\right)\right\} \\
&\text{subject to} && x_1+x_2+\cdots+x_L=X_T \\
& && x_i\geq 0 \qquad \forall i=1,2,\cdots,L
\end{aligned}$$

By using the Method of Lagrangian Multiplier,

Let $f(\underline{\mathbf{x}})=-\dfrac{1}{L}\log\left\{\prod_{j=1}^{L}\left[1+\dfrac{(L+1)x_j}{N_o}\right]\cdot\left(1-\sum_{i=1}^{L}\dfrac{x_i}{N_o+(L+1)x_i}\right)\right\}$, $\underline{\mathbf{x}}=[x_1,x_2,\cdots,x_L]^T$

$$\nabla f(\underline{\mathbf{x}}) = \begin{bmatrix} \frac{\partial f(\underline{\mathbf{x}})}{\partial x_1} \\ \frac{\partial f(\underline{\mathbf{x}})}{\partial x_2} \\ \vdots \\ \frac{\partial f(\underline{\mathbf{x}})}{\partial x_L} \end{bmatrix} \quad \text{and} \quad \nabla h(\underline{\mathbf{x}}) = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

where

$$\frac{\partial f(\underline{\mathbf{x}})}{\partial x_m} = -\frac{L+1}{N_o L + L(L+1)x_m} + \frac{N_o}{L\left[1 - \sum_{i=1}^{L} \frac{x_i}{N_o + (L+1)x_i}\right][N_o + (L+1)x_m]^2}$$

Let $\underline{\mathbf{x}}^*$ be the local minimum of f.
By the first order necessary condition:

$$\nabla f(\underline{\mathbf{x}}^*) + \lambda^* \nabla h(\underline{\mathbf{x}}) = 0$$

$$\Rightarrow \begin{cases} \frac{\partial f(\underline{\mathbf{x}}^*)}{\partial x_1} + \lambda^* = 0 \\ \frac{\partial f(\underline{\mathbf{x}}^*)}{\partial x_2} + \lambda^* = 0 \\ \vdots \\ \frac{\partial f(\underline{\mathbf{x}}^*)}{\partial x_L} + \lambda^* = 0 \end{cases}$$

It can be easily checked that

$$x_1^* = x_2^* = \cdots = x_L^* = \frac{X_T}{L}$$

is the solution of above L equations.

The Hessian of the Lagrangian is:

$$
\begin{aligned}
\nabla^2_{xx} L(\underline{\mathbf{x}}^*, \lambda^*) &= \nabla^2 f(\underline{\mathbf{x}}^*) + \lambda^* \nabla^2 h(\underline{\mathbf{x}}^*) \\
&= \nabla^2 f(\underline{\mathbf{x}}^*) \qquad\qquad\qquad\qquad \text{since } \nabla^2 h(\underline{\mathbf{x}}^*) = \mathbf{0} \\
&= \begin{pmatrix}
\frac{\partial^2 f(\underline{\mathbf{x}}^*)}{\partial x_1^{\ 2}} & \frac{\partial^2 f(\underline{\mathbf{x}}^*)}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f(\underline{\mathbf{x}}^*)}{\partial x_1 \partial x_L} \\
\frac{\partial^2 f(\underline{\mathbf{x}}^*)}{\partial x_2 \partial x_1} & \frac{\partial^2 f(\underline{\mathbf{x}}^*)}{\partial x_2^{\ 2}} & & \frac{\partial^2 f(\underline{\mathbf{x}}^*)}{\partial x_2 \partial x_L} \\
 & & \ddots & \vdots \\
\frac{\partial^2 f(\underline{\mathbf{x}}^*)}{\partial x_L \partial x_1} & \frac{\partial^2 f(\underline{\mathbf{x}}^*)}{\partial x_L \partial x_2} & \cdots & \frac{\partial^2 f(\underline{\mathbf{x}}^*)}{\partial x_L^{\ 2}}
\end{pmatrix}
\end{aligned}
$$

By the symmetry of the first order necessary conditions and $\underline{\mathbf{x}}^*$, we have

$$
\frac{\partial^2 f(\underline{\mathbf{x}}^*)}{\partial x_m^{\ 2}} = \frac{\partial^2 f(\underline{\mathbf{x}}^*)}{\partial x_n^{\ 2}} \qquad \forall m, n
$$

$$
\frac{\partial^2 f(\underline{\mathbf{x}}^*)}{\partial x_m \partial x_n} = \frac{\partial^2 f(\underline{\mathbf{x}}^*)}{\partial x_o \partial x_p} \qquad \forall i \neq k, o \neq p
$$

Let $a = \frac{\partial^2 f(\underline{\mathbf{x}}^*)}{\partial x_m^{\ 2}}$ $\qquad \forall m = 1, 2, \cdots, L$

$b = \frac{\partial^2 f(\underline{\mathbf{x}}^*)}{\partial x_m \partial x_n}$ $\qquad \forall m \neq n$

$$
\nabla^2_{xx} L(\underline{\mathbf{x}}^*, \lambda^*) = \begin{pmatrix} a & b & \cdots & b \\ b & a & & b \\ \vdots & & \ddots & \vdots \\ b & b & \cdots & a \end{pmatrix}
$$

To test whether the point $\underline{\mathbf{x}}^* = \left(\frac{X_T}{L}, \frac{X_T}{L}, \cdots, \frac{X_T}{L} \right)$ is a strict local minimum

Consider

$$
\forall \underline{\mathbf{y}} \in V = \left\{ \underline{\mathbf{y}} \mid \nabla h(\underline{\mathbf{x}})^T \underline{\mathbf{y}} = 0 \right\} = \left\{ \underline{\mathbf{y}} \mid y_1 + y_2 + \cdots + y_L = 0 \right\} \text{ with } \underline{\mathbf{y}} \neq 0
$$

$$\underline{\mathbf{y}}^T \nabla_{xx}^2 L(\underline{\mathbf{x}}^*, \lambda^*)\underline{\mathbf{y}} = \underline{\mathbf{y}}^T \begin{pmatrix} a & b & \cdots & b \\ b & a & & b \\ \vdots & & \ddots & \vdots \\ b & b & \cdots & a \end{pmatrix} \underline{\mathbf{y}}$$

$$= \underline{\mathbf{y}}^T \left[\begin{pmatrix} b & b & \cdots & b \\ b & b & & b \\ \vdots & & \ddots & \vdots \\ b & b & \cdots & b \end{pmatrix} + \begin{pmatrix} a-b & 0 & \cdots & 0 \\ 0 & a-b & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & a-b \end{pmatrix} \right] \underline{\mathbf{y}}$$

$$= \underline{\mathbf{y}}^T \begin{pmatrix} b & b & \cdots & b \\ b & b & & b \\ \vdots & & \ddots & \vdots \\ b & b & \cdots & b \end{pmatrix} \underline{\mathbf{y}} + \underline{\mathbf{y}}^T \begin{pmatrix} a-b & 0 & \cdots & 0 \\ 0 & a-b & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & a-b \end{pmatrix} \underline{\mathbf{y}}$$

$$= \underline{\mathbf{y}}^T \begin{pmatrix} a-b & 0 & \cdots & 0 \\ 0 & a-b & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & a-b \end{pmatrix} \underline{\mathbf{y}} \qquad \text{since } y_1 + y_2 + \cdots + y_L = 0$$

$$= (a-b)(y_1^2 + y_2^2 + \cdots + y_L^2)$$

so,

$$\underline{\mathbf{y}}^T \nabla_{xx}^2 L(\underline{\mathbf{x}}^*, \lambda^*)\underline{\mathbf{y}} > 0 \qquad \text{iff} \qquad (a-b) > 0$$

Consider

$$b = \frac{\partial^2 f(\underline{\mathbf{x}}^*)}{\partial x_m \partial x_n}$$

$$= \frac{\partial}{\partial x_n} \left\{ -\frac{L+1}{N_o L + L(L+1)x_m} + \frac{N_o}{L\left[1 - \sum_{i=1}^{L} \frac{x_i}{N_o + (L+1)x_i}\right]\left[N_o + (L+1)x_m\right]^2} \right\}$$

$$= \frac{N_o^{\,2}}{L\left[N_o + (L+1)x_m\right]^2 \left[N_o + (L+1)x_n\right]^2 \left[1 - \sum_{i=1}^{L} \frac{x_i}{N_o + (L+1)x_i}\right]^2}$$

$$b = \frac{N_o^{\,2}}{L\left[N_o + \frac{L+1}{L}P_T\right]^4 \left[1 - \frac{P_T}{N_o + \frac{L+1}{L}P_T}\right]^2}$$

$$a = \frac{\partial^2 f(\underline{\mathbf{x}}^*)}{\partial x_m{}^2}$$

$$= \frac{\partial}{\partial x_m}\left\{-\frac{L+1}{N_o L + L(L+1)x_m} + \frac{N_o}{L\left[1-\sum_{i=1}^{L}\frac{x_i}{N_o+(L+1)x_i}\right]\left[N_o+(L+1)x_m\right]^2}\right\}$$

$$= \frac{L(L+1)^2}{\left[N_o L + L(L+1)x_m\right]^2} - \frac{2N_0(L+1)}{L\left[N_o+(L+1)x_m\right]^3\left[1-\sum_{i=1}^{L}\frac{x_i}{N_o+(L+1)x_i}\right]} +$$

$$\frac{N_o{}^2}{L\left[N_o+(L+1)x_m\right]^4\left[1-\sum_{i=1}^{L}\frac{x_i}{N_o+(L+1)x_i}\right]^2}$$

$$a = \frac{L(L+1)^2}{\left[N_o L + (L+1)P_T\right]^2} - \frac{2N_0(L+1)}{L\left[N_o+\frac{L+1}{L}P_T\right]^3\left[1-\frac{P_T}{N_o+\frac{L+1}{L}P_T}\right]} +$$

$$\frac{N_o{}^2}{L\left[N_o+\frac{L+1}{L}P_T\right]^4\left[1-\frac{P_T}{N_o+\frac{L+1}{L}P_T}\right]^2}$$

$$
\begin{aligned}
a-b &= \frac{L(L+1)^2}{\left[N_o L+(L+1)P_T\right]^2} - \frac{2N_0(L+1)}{L\left[N_o+\dfrac{L+1}{L}P_T\right]^3\left[1-\dfrac{P_T}{N_o+\dfrac{L+1}{L}P_T}\right]} \\
&= \frac{(L+1)^2}{L\left[N_o+\dfrac{L+1}{L}P_T\right]^2} - \frac{2N_0(L+1)}{L\left[N_o+\dfrac{L+1}{L}P_T\right]^2\left[N_o+\dfrac{L+1}{L}P_T-P_T\right]} \\
&= \frac{(L+1)}{L\left[N_o+\dfrac{L+1}{L}P_T\right]^2}\left\{(L+1)-\frac{2N_0}{\left[N_o+\dfrac{1}{L}P_T\right]}\right\} \\
&= \frac{(L+1)}{L\left[N_o+\dfrac{L+1}{L}P_T\right]^2}\left\{\frac{(L-1)N_o+\dfrac{L+1}{L}P_T}{\left[N_o+\dfrac{1}{L}P_T\right]}\right\} \\
&> 0
\end{aligned}
$$

Therefore, the point $\underline{\mathbf{x}}^* = \left(\frac{X_T}{L}, \frac{X_T}{L}, \cdots, \frac{X_T}{L}\right)$ is a strict local minimum.

By changing the variables back from x_j to $P_{(j)}$, X_T to P_{tot}, and substitute the optimal solution to the sum capacity formula, the result follows.

Lemma 4:

$$f(\underline{\mathbf{x}})=\prod_{j=1}^{L}\left[1+\frac{(L+1)x_j}{N_o}\right]\cdot\left(1-\sum_{i=1}^{L}\frac{x_i}{N_o+(L+1)x_i}\right)$$ is an increasing function of $\underline{\mathbf{x}}$,

Proof of Lemma 4:

Let $f(\underline{\mathbf{x}})=\prod_{j=1}^{L}\left[1+\frac{(L+1)x_j}{N_o}\right]\cdot\left(1-\sum_{i=1}^{L}\frac{x_i}{N_o+(L+1)x_i}\right)$

$$\begin{aligned}\frac{\partial f(\underline{\mathbf{x}})}{\partial x_m}&=\prod_{j=1}^{L}\left[1+\frac{(L+1)x_j}{N_o}\right]\cdot\left[-\frac{N_o}{\left(N_o+(L+1)x_m\right)^2}\right]\\&\quad+\prod_{\substack{j=1\\j\neq m}}^{L}\left[1+\frac{(L+1)x_j}{N_o}\right]\cdot\left(\frac{L+1}{N_o}\right)\cdot\left(1-\sum_{i=1}^{L}\frac{x_i}{N_o+(L+1)x_i}\right)\\&=\prod_{\substack{j=1\\j\neq m}}^{L}\left[1+\frac{(L+1)x_j}{N_o}\right]\cdot\left\{\left[1+\frac{(L+1)x_m}{N_o}\right]\cdot\left[-\frac{N_o}{\left(N_o+(L+1)x_m\right)^2}\right]\right.\\&\quad\left.+\left(\frac{L+1}{N_o}\right)\cdot\left(1-\sum_{i=1}^{L}\frac{x_i}{N_o+(L+1)x_i}\right)\right\}\\&=\prod_{\substack{j=1\\j\neq m}}^{L}\left[1+\frac{(L+1)x_j}{N_o}\right]\cdot\left\{-\frac{1}{N_o+(L+1)x_m}+\left(\frac{L+1}{N_o}\right)\cdot\left(1-\sum_{i=1}^{L}\frac{x_i}{N_o+(L+1)x_i}\right)\right\}\end{aligned}$$

Since $N_o\geq 0$, therefore

$$\begin{aligned}1-\sum_{i=1}^{L}\frac{x_i}{N_o+(L+1)x_i}&\geq 1-\sum_{i=1}^{L}\frac{x_i}{(L+1)x_i}\\&=1-\sum_{i=1}^{L}\frac{1}{(L+1)}\\&=\frac{1}{L+1}\end{aligned}$$

$$\frac{\partial f(\underline{\mathbf{x}})}{\partial x_m}\geq\prod_{\substack{j=1\\j\neq m}}^{L}\left[1+\frac{(L+1)x_j}{N_o}\right]\cdot\left\{-\frac{1}{N_o+(L+1)x_m}+\left(\frac{L+1}{N_o}\right)\cdot\left(\frac{1}{L+1}\right)\right\}$$

$$= \prod_{\substack{j=1 \\ j \neq m}}^{L} \left[1 + \frac{(L+1)x_j}{N_o} \right] \cdot \left\{ \frac{1}{N_o} - \frac{1}{N_o + (L+1)x_m} \right\}$$

$$= \prod_{\substack{j=1 \\ j \neq m}}^{L} \left[1 + \frac{(L+1)x_j}{N_o} \right] \cdot \left\{ \frac{(L+1)x_m}{N_o\left(N_o + (L+1)x_m\right)} \right\}$$

$$\geq 0$$

Therefore $f(\underline{\mathbf{x}})$ is an increasing function of $\underline{\mathbf{x}}$.

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